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RESEARCH ARTICLE

Stag Hunt with unknown outside options

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Abstract
We study the Stag Hunt game where two players simultaneously decide whether to cooperate or to choose their outside options (defect). A player's gain from defection is his private information (the type). The two players' types are independently drawn from the same cumulative distribution. We focus on the case where only a small proportion of types are dominant (higher than the value from cooperation). It is shown that for a wide family of distribution functions, if the players interact only once, the unique equilibrium outcome is defection by all types of player. Whereas if a second interaction is possible, the players will cooperate with positive probability and already in the first period. Further restricting the family of distributions to those that are sufficiently close to the uniform distribution, cooperation in both period with probability close to 1 is achieved, and this is true even if the probability of a second interaction is very small.

Keywords Stag Hunt · Coordination · Private information · Repeated interaction

JEL Classification C72 · C73 · D82 · D83

1 Introduction

Two players consider a cooperation to execute a joint project. They simultaneously decide on whether or not to cooperate. If both players cooperate, the project will be successfully complete and each one of them will obtain 1. If only one player cooperates, he wastes his effort and receives 0. A player who decides not to cooperate (defects) obtains a certain payoff from his outside option, which might be slightly higher if

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the other player cooperates rather than defects.[1] The complete information case is a version of the well known Stag Hunt game. If the payoffs from the outside options are not too large, it has two pure strategy Nash equilibria: both players cooperate, or both players defect. The former is payoff dominant, and the latter is risk dominant if their outside options are not too small.

In the case where the values of the outside options are private information, the payoff dominant outcome is sensitive to small perturbations. The introduction of even small uncertainty may eliminate the cooperative outcome. For example, suppose that each player's payoff from the outside option is independently drawn from a uniform distribution on $[0, B]$, where $B > 1$. Namely, a positive proportion of types (the "dominant types") have values higher than 1, and it is their strictly dominant strategy to defect. In this case, defection by *all* types of players is the unique Bayesian-Nash equilibrium, irrespective of how small the size of dominant types is.

Intuitively, the existence of dominant types of one player induces the non-dominant types of the other player, whose outside option is close to 1, to defect (the probability of cooperation is less than 1 and a player with an outside option sufficiently close to 1 from below is better off defecting). This further induces players with even lower outside options to defect. When types are drawn, for instance, from the uniform distribution, this creates "an escalating cycle of pessimistic expectations that spiral toward the Pareto inferior equilibrium" (see Schelling 1960; Morris and Shin 2003; and Baliga and Sjöström 2004).

The condition on type distributions that guarantees full defection as the unique equilibrium is typically referred to as the "multiplier condition". It requires that for any $d \in (0, 1]$, the probability that a player is of type at most d is smaller than d (that is, $F(d) < d$). When the multiplier condition is satisfied, can the players escape the full defection outcome? One affirmative answer is that they can through informative cheap-talk (see Baliga and Sjöström 2004). But informative cheap-talk may not always be implementable. It may, for instance, be prohibited by (anti-trust) law or by regulation.

In this paper, we show that even in the absence of cheap-talk, the pessimistic circle can still be avoided, if the two players after playing the game once, may play with a positive probability the same game once again (even if this probability is small).

In the perfect Bayesian equilibrium highlighted in this paper, a player with a type below some threshold d^* behaves trustingly. He cooperates in the first period and continues to cooperate in the second period, regardless of the first period action of his counterpart (the first period cooperation by a player serves as a signal for being a non-dominant type, in the hope that the other player will cooperate in the second period). A player with a type above d^* but below 1 (non-dominant type) behaves cautiously. He defects in the first period, and mimics the counterpart's first period behaviour in the second period. A player with a type above 1 (dominant type) always defects. In this equilibrium, any type that cooperates in the first period continues to cooperate in the second period. This by itself initiates an escalation of positive expectations.

Indeed, in this case, a player of a non-dominant type is better off cooperating in the second period, if his counterpart cooperates in the first period. When the size of

[1] The assumption that the defecting player slightly prefers his counterpart to cooperate fits an arms race scenario, where a player who decides to build a new weapon is better off when his counterpart refrains from doing the same (see e.g., Baliga and Sjöström 2004).

dominant types of the two players are positive but small, a unilateral cooperation initiates with high probability a mutual cooperation in the second period. This motivates player i with a small outside option (low type) to cooperate in the first period, even if he believes that with probability 1 the other player j defects in this period. Since the same argument applies to both players, j cooperates with a positive probability already in the first period. Taking this into account, player i updates his belief on j's first period action, and now slightly higher types of i will also cooperate in the first period, and so on. For the uniform distribution with a sufficiently small proportion of dominant types, this optimistic escalation continues until almost full cooperation is achieved in the first period.[2] This is in a drastic contrast to the one-shot game, where the unique equilibrium is defection by all types of the two players.

We show that in addition to the uniform distribution, the above observation holds true for distributions satisfying on one hand the multiplier condition, and on the other hand are sufficiently close to the uniform distribution. For these distributions, as the proportion of dominant types becomes sufficiently small, (i) in the one-shot game, full defection is the unique Bayesian-Nash equilibrium, (ii) if the players may interact twice, with probability close to 1 cooperation can be attained already in the first period.

Let us provide some intuition on why for (ii) we need to deal with distributions that are close to the uniform distribution. Namely, for what distributions the optimistic escalation continues until 1. Suppose the escalation ends up with a first period threshold $d^* \in (0, 1)$. Now consider Player 1 with type d_1 slightly higher than d^*. If he cooperates rather than defects, he loses $d_1 - F(d^*)$ in the first period,[3] while he gains in the second period from Player 2's positive reaction to 1's cooperative signal. The first period loss $d_1 - F(d^*)$ is very close to $d^* - F(d^*)$, for d_1 very close to d^* from above. If F is sufficiently close to the uniform distribution, then by reducing the proportion of dominant types, $d^* - F(d^*)$ approaches zero, and hence the first period loss can also be made sufficiently small. The second period benefit, however, does not shrink to zero as the distribution approaches the uniform distribution on [0, 1], and hence it exceeds the first period loss. Consequently, with sufficiently small proportion of dominant types ($B \downarrow 1$), players with types slightly higher than d^* cooperates. The above argument applies to every $d^* < 1$, and hence the first period cooperative threshold increases to 1 as $B \downarrow 1$. A more detailed argument is provided in the paragraphs following Proposition 4.

Related literature: It has been argued in the literature (e.g., Angeletos et al. 2007) that when full defection is the unique equilibrium of the static game, the information in the repeated game is revealed endogenously through the first period action and it generates an update of priors that may admit multiple equilibria in the second stage. This, however, can explain only the cooperations from period 2 onward. Indeed, in Angeletos et al. (2007), when the game changes from static to dynamic, the players' first period actions remain unchanged. This is not the case in our paper. If the players know they may meet with positive probability for a second time, almost-full cooper-

[2] For the uniform distributions over $[0, B]$ where $B > 1$, we actually characterize *all* symmetric equilibria of the two-period interaction game and show that except for the full defection equilibrium, all other equilibria exhibit almost-full cooperation, and already in the first period, when $B \downarrow 1$.

[3] $d_1 > d^* > F(d^*)$ by the multiplier condition.

ation is achieved and already in the first period, even if the probability of a second meeting is very small. The endogenous revelation of information through past play is therefore not sufficient to explain our result.

Our model is closely related to the arms race model of Baliga and Sjöström (2004), thereafter BS. Both papers study a coordination game with two-sided private information, and show that for some distributions of types, if there is a small probability that a player has a dominant type (prefers to defect irrespective of the other player's action), the unique equilibrium of the one-shot game has defecting with probability one. BS further shows that if communication is allowed, there are cheap-talk equilibria with informative talks that induces almost-full cooperation.

The main difference between our model and BS is the replacement of cheap-talk with a two-period interaction. This leads to entirely different equilibrium structures. In BS, signals are sent through costless cheap-talk. To attain a positive cooperation, the equilibrium is necessarily "non-monotonic" in the sense that the equilibrium strategy is defined by two or more different thresholds. BS show that if, instead, the cheap-talk is monotonic, then full defection is the unique equilibrium outcome. In our model, sending signals through actions is costly and such non-monotonicity does not occur. In fact, any equilibrium strategy of our two-period game must be monotonic in the sense that there is a threshold below which all types cooperate and above which all types defect (see Remark 1 for a detailed discussion).

Our results differ significantly from the literature on "gradualism", which studies how partners who are uncertain about each other's types can achieve cooperation in an infinitely repeated prisoner's dilemma game (Sobel 1985; Ghosh and Ray 1996; Watson 1999; Furusawa and Kawakami 2008). The equilibrium there has the property of "gradual trust-building", i.e., partners start with a low level of cooperation and gradually increase it as the initial phases are successfully passed without defection. Kandori (1992), Ellison (1994), Takahashi (2010) and Heller and Mohlin (2018) also study how to sustain cooperation between partners who are engaged in an infinitely repeated interaction. In these papers players are randomly matched with new partners in each period. Huang (2018) studies a related model on coordination and social learning with short-lived players.

The paper is organized as follows. Section 2 presents the model and studies conditions under which full defection is the unique equilibrium in the one-shot game, while almost-full cooperation can be attained in both periods as a perfect Bayesian equilibrium of the two period game. Section 3 provides a complete analysis of the uniform distribution case. Section 4 extends the analysis to games with more than 2 periods. Conclusion and discussion are the subject of Sect. 5. Proofs are relegated to the "Appendix".

2 The model

2.1 The one-shot game

Consider the following one-shot Bayesian game. There are two players 1 and 2. Each player has two strategies: C (cooperate) and D (defect). The actions are taken simul-

Table 1 The payoff structure

1	2	
	D	C
D	d_1, d_2	$d_1 + \mu, 0$
C	$0, d_2 + \mu$	$1, 1$

taneously and the payoff structure is given in Table 1. It is assumed that d_i is a private information of player i, whereas $\mu \geq 0$ is commonly known. The types d_1 and d_2 are independently drawn from the same distribution, with continuous cumulative distribution function F. F has support $[0, B]$ with $F(0) = 0$, $F(d)$ strictly increasing for every $d \in (0, B)$, and $F(B) = 1$. Except for players' realized types d_1 and d_2, all the above is common knowledge.

Here d_i represents the payoff player i can guarantee to obtain by choosing his outside option, irrespective of j's action. While μ is a player's additional satisfaction if the other player switches from defecting to cooperate, while the player himself defects. The case $\mu > 0$ fits scenario like arms race, where a player who defects and builds a superior weapon has a lead over his counterpart who acts cooperatively and does not build a new weapon. If $d_i > 1 - \mu$, defection is a strictly dominant strategy for player i, and he defects regardless of the counterpart's action. We therefore refer to types above $1 - \mu$ as "dominant types".

A *pure strategy* s_i of player i is a Borel-measurable function $s_i : [0, B] \to \{C, D\}$. Namely, a choice of action for every type d_i of player i so that $s_i^{-1}(C)$ is a Borel subset of $[0, B]$. The above describes a game $G_1(F, \mu)$. A pair of pure strategies $s^* = (s_1^*, s_2^*)$ is a *Bayesian (Nash) equilibrium* of $G_1(F, \mu)$ iff for each $d_i \in [0, B]$ the action $s_i^*(d_i)$ maximizes the expected payoff of player i of type d_i, given s_j^*, $j \neq i$.

Let $d \in [0, B]$ and let s_i^d be a threshold strategy. That is, for $d > 0$,

$$s_i^d = \begin{cases} C & \text{if } 0 \leq d_i < d \\ D & \text{if } d < d_i \leq B \\ \in \{C, D\} & \text{if } d_i = d, \end{cases} \tag{1}$$

and for $d = 0$,

$$s_i^0 = \begin{cases} C & \text{if } d_i = 0 \\ D & \text{if } 0 < d_i \leq B. \end{cases} \tag{2}$$

Since player i of type $d_i > 0$ is best off defecting if the other player j chooses s_j^0, the full defection strategy (s_1^0, s_2^0) is always an equilibrium of $G_1(F, \mu)$, regardless of μ and F. The next proposition characterizes all equilibria of $G_1(F, \mu)$.

Proposition 1 *Let $F : [0, B] \to [0, 1]$ be a continuous distribution function.*

(i) Every equilibrium of $G_1(F, \mu)$ consists of threshold strategies.
(ii) If $B > 1 - \mu$, then s^* is an equilibrium of $G_1(F, \mu)$ iff $s_i^* = s_i^d$, where d is a fixed point of $(1 - \mu)F$.

Proof See Sect. A.1 of the "Appendix". □

To sharpen the difference between a one-shot interaction and a twice repeated interaction of the players, we focus on type distributions under which full defection is the unique equilibrium of the one-shot game. This is guaranteed by the following multiplier condition, first proposed by Baliga and Sjöström (2004).

Definition 1 The distribution function F satisfies the *multiplier condition* if $F(d) < d$ for all $d \in (0, 1]$.

If F satisfies the multiplier condition, then for any $\mu \geq 0$, we have $(1-\mu)F(d) < d$ for all $d \in (0, 1]$, and hence by Proposition 1, full defection is the unique equilibrium of $G_1(F, \mu)$. This leads to the following corollary, which resembles Theorem 1 in Baliga and Sjöström (2004).

Corollary 1 *If F satisfies the multiplier condition, then for any $\mu \geq 0$, full defection is the unique Bayesian-Nash equilibrium for $G_1(F, \mu)$.*

Intuitively, a dominant type of both players (types above $1-\mu$) certainly defects. Knowing that the other player defects with positive probability, a type that is "almost" a dominant type (types lower than but close to $1-\mu$) also defects. This induces players with a even lower type to defect, and so on. If $(1-\mu)F$ has a fixed point $d > 0$, then the contagion stops at d. All types below d cooperate and all types above d defect. The multiplier condition however implies that $(1-\mu)F$ has no fixed point except 0. Hence the contagion continues and as a result all types (except 0) defect. See Baliga and Sjöström (2004) for a more detailed discussion on this effect.

2.2 The two-period game

In this section we analyze the two-period game $G_2(F, \mu)$. The players in $G_2(F, \mu)$ play the game $G_1(F, \mu)$ twice. It is assumed that the types of the players do not change across periods. The actions of the players are publicly observed after each period. The payoff of player i with type d_i is the discounted sum of his expected payoffs in every period. Let $0 < \delta < 1$ be the discount rate (δ can also be interpreted as the probability of players meeting twice).

An action of player i in period t, $t = 1, 2$, is a Borel-measurable function from $[0, B] \times H^{t-1}$ to $\{C, D\}$, where H^{t-1} is the set of histories up to period $t-1$ with the convention of $H^0 = \{\emptyset\}$. That is, the history at the start of the game is $\emptyset$ and it contains no action. The set of histories after the first period is

$$H^1 = \{(C, C), (C, D), (D, C), (D, D)\}.$$

Let $\mathscr{H} = \{\emptyset\} \cup H^1$. A *pure strategy* of player i in the two-period game is a Borel-measurable function s_i,

$$s_i : [0, B] \times \mathscr{H} \to \{C, D\}.$$

A *belief system* of a player consists of beliefs the player has about the other player's type at any information set. A pair of pure strategies $s^* = (s_1^*, s_2^*)$ and a pair of belief systems $b^* = (b_1^*, b_2^*)$ constitute a *Perfect Bayesian Equilibrium* (PBE) of $G_2(F, \mu)$ if and only (i) for every type $d_i \in [0, B]$ and every history $h \in \mathscr{H}$, the strategy $s_i^*(d_i, h)$ maximizes the expected payoff of i, given his belief b_i^*, and (ii) b^* is computed from s^* using Bayes' rule wherever possible (the beliefs off-the-equilibrium can be arbitrary). We sometimes omit the belief system if the Bayes' rule applies everywhere.

Let $\gamma_i := (d_i^{\emptyset}, d_i^{C,C}, d_i^{C,D}, d_i^{D,C}, d_i^{D,D}) \in \mathbb{R}_+^5$, $i = 1, 2$. Every γ_i defines a threshold strategy s_i of i by

$$s_i(d_i, h) = C \text{ iff } 0 \le d_i \le d_i^h, \quad h \in \mathscr{H}. \tag{3}$$

Like the one-shot game, full defection in both periods is an equilibrium outcome of $G_2(F, \mu)$ for all distribution functions F on $[0, B]$. Nevertheless, this equilibrium is always interim Pareto dominated by all other equilibria (i.e., not only the ex-ante payoff of full defection is lower, any type is doing worse in full defection than in any other equilibrium). The next proposition asserts that all equilibrium strategies of $G_2(F, \mu)$ are threshold strategies.

Proposition 2 *Let* $F : [0, B] \to [0, 1]$*,* $B > 1 - \mu$*, be a continuous and strictly increasing distribution function. Let* $s^* = (s_1^*, s_2^*)$ *be an equilibrium of* $G_2(F, \mu)$*. Then* s_1^* *and* s_2^* *are threshold strategies.*

Proof See Sect. A.2 of the "Appendix". □

By Proposition 2, we only consider threshold strategies as candidates for equilibrium. The next proposition focuses on games under which full defection is the unique equilibrium of the one-shot game, and shows that for sufficiently small μ, the two-period game $G_2(F, \mu)$ has a PBE where both players cooperate with positive probability in both periods.

Proposition 3 *Let* $F : [0, B] \to [0, 1]$ *be a continuous and strictly increasing distribution function. Suppose* $B > 1$ *and the multiplier condition is satisfied. Then there exists* $\widehat{\mu} > 0$ *such that for every* $\mu \in [0, \widehat{\mu})$*, the game* $G_2(F, \mu)$ *has a PBE with* $d^* > 0$ *such that a player cooperates in the first period iff his type does not exceed* d^**. Moreover, a player with type below* d^* *continues to cooperate in the second period, regardless of the other player's first period action.*

Proof See Sect. A.3 of the "Appendix". □

If the multiplier condition is satisfied, then full defection is the unique equilibrium of the one-shot game $G_1(F, \mu)$ for every $\mu \ge 0$ (see Corollary 1). Proposition 3 states that cooperation in $G_2(F, \mu)$ can be achieved with positive probability under the same condition.

It is shown in "Appendix A.3" that under the conditions of Proposition 3, the following pair of strategies $s^* = (s_1^*, s_2^*)$ is an equilibrium of $G_2(F, \mu)$, if μ is sufficiently small.

$$s_1^* = (d_1^{\emptyset} = d^*, d_1^{C,D} = m^*, d_1^{C,C} = d_1^{D,C} = 1 - \mu \text{ and } d_1^{D,D} = 0), \tag{4}$$

$$s_2^* = (d_2^{\emptyset} = d^*, d_2^{D,C} = m^*, d_2^{C,C} = d_2^{C,D} = 1-\mu \text{ and } d_2^{D,D} = 0), \tag{5}$$

where $d^* < m^* < 1-\mu$. The first-period threshold d^* is a solution d in $(0, 1-\mu)$ to[4]

$$d = \frac{(1-\delta-\mu)F(d) + \delta F(1-\mu)}{1+\delta-\delta F(d)}, \tag{6}$$

and m^* is defined by

$$m^* := (1-\mu) \cdot \frac{F(1-\mu) - F(d^*)}{1-F(d^*)}. \tag{7}$$

In equilibrium s^*, player i who cooperates in the first period is of type d_i, $d_i \leq d^* < m^*$, and hence he continues to cooperate in the second period, regardless of the other player's first period action.[5] The first period cooperation is a credible signal of player i's (non-dominant) type, and it induces all non-dominant types of the other player j to cooperate in the second period, and irrespective of player j's first-period action. Consequently, under s^*, both players cooperate in the second period unless both of them defect in the first period.

For our next result, let F_1 be a distribution function on [0, 1], and let $\mathscr{F}(F_1)$ be the set of all distribution functions F_B, which is obtained by rescaling the units of F_1. Formally, $F_B \in \mathscr{F}(F_1)$ if and only if $F_B(x) = F_1(\frac{x}{B})$ for every $x \in [0, B]$. For $B > 1$, the larger B is, the higher is the proportion of dominant types (types $d \in [1-\mu, B]$).

Definition 2 Let $\mathcal{G}_1$ be the set of all continuous and strictly increasing distribution functions $F_1 : [0, 1] \to [0, 1]$ such that $F_1(d) \leq d$ for all $d \in [0, 1]$.

Let $F_1 \in \mathcal{G}_1$. Then F_1 first-order stochastically dominates (FOSD) the uniform distribution on [0, 1]. If $F_B \in \mathscr{F}(F_1)$ and $B > 1$, then the multiplier condition $F_B(d) < d$, $d \in (0, 1]$ is satisfied and Proposition 3 applies. The next proposition offers sufficient conditions for d^* in s^* to be sufficiently close to 1 (that is, almost-full cooperation is attained in s^* in both periods).

Proposition 4 *Let* $\delta \in (0, 1)$. *Suppose*

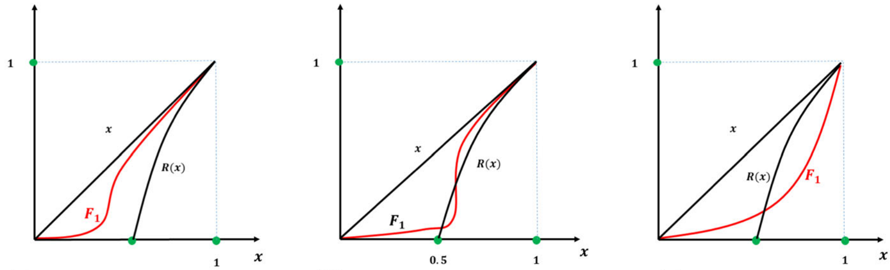
(i) $F_1 \in \mathcal{G}_1$, *and*

$$\text{(ii) there exists } \widehat{x} \in [0,1) \text{ s.t. } F_1(x) > \overbrace{\max\left(0, \frac{x-\delta(1-x)}{1-\delta(1-x)}\right)}^{R(x):=} \text{ for all } x \in (\widehat{x}, 1). \tag{8}$$

Then for every $\epsilon > 0$, *there exists* $\mu' > 0$ *and* $\eta > 0$ *such that for every* $\mu \in [0, \mu')$, $B \in (1, 1+\eta)$, *and* $F_B \in \mathscr{F}(F_1)$ *the following holds:*

[4] Equation (6) can have multiple solutions of d in $(0, 1-\mu)$. If $\mu = 0$, the first period threshold d^* can take the value of any solution (in particular, the maximal one). If $\mu > 0$, then the selection of the solutions is more complicated, and the detail is provided in "Appendix A.3".

[5] Note that $d_1^{C,D} = m^*$ is a second period threshold for cooperation after (C, D) whether or not Player 1 follows s^* in the first period. Player 1 of type $d_1 \in (d^*, m^*]$ should defect in period 1 according to s^*, but off-equilibrium if he cooperates in the first period, he is best off cooperating also in the second period.



(a) F_1 satisfies Proposition 3 for $\hat{x} = 0$ **(b)** F_1 satisfies Proposition 3 for $\hat{x} > 0$ **(c)** F_1 violates Proposition 3

Fig. 1 Functions that (don't) satisfy Proposition 4 ($\delta = 0.9$)

(i) The full defection equilibrium is the unique equilibrium of the one-shot game $G_1(F_B, \mu)$.

(ii) In the two-period game $G_2(F_B, \mu)$, s^* *defined in* (4)–(5) *is a PBE and* $d^* \in (1 - \epsilon, 1 - \mu)$.

Proof See Sect. A.4 of the "Appendix". □

Proposition 4 provides a class of distributions F_1 that FOSD the uniform distribution ($F_1 \in \mathcal{G}_1$) and is sufficiently close to the uniform distribution for high types (see (8) and Fig. 1). Let F_1 be in this class. If F_B, a re-scaling of F_1, has a positive, but sufficiently small proportion of types above 1 (i.e., if $B \downarrow 1$), and if μ is sufficiently small, then by Proposition 4, while full defection is the unique equilibrium of $G_1(F_B, \mu)$, almost-full cooperation in both periods is attained as a PBE of $G_2(F_B, \mu)$. We next provide some intuition for this result.

According to the equilibrium s^*, a player who cooperates in the first period continues to cooperate in the second period. Hence, a player of a non-dominant type is better off cooperating in the second period if his counterpart cooperates in the first period. Taking this into account, a player with a sufficiently low outside option prefers to cooperate in the first period, even if he believes that in this period his counterpart will not cooperate. Therefore, both players assign a positive probability to the event that their counterparts will cooperate already in the first period. Taking this into account, players with a slightly higher outside option will also cooperate in the first period, and so on.

This escalation of positive expectations continues until some cut-off type $d^* \in (0, 1 - \mu)$. Below d^* all types cooperate and above d^* all types defect. Suppose the type distribution is sufficiently close to the uniform distribution (condition (8)), then the contagion continues all the way to 1 as $B \downarrow 1$. We next elaborate on why we need for this result condition (8) on type distributions. For simplicity let $\mu = 0$ and let us deal with distribution functions that satisfy (8) for $\widehat{x} = 0$ (see Fig. 1a).

In s^*, a player signals that he is a non-dominant type by cooperating in the first period. The signalling cost of the cut-off type d^* is $d^* - F(d^*)$. Given that F is close to the uniform distribution on [0, 1], the cut-off type's signalling cost is close to zero, regardless of the value of d^*. The benefit of a first period cooperation, on the other hand, comes from changing the second-period outcome from (D, D) to (C, C)

against medium type counterpart (types above d^* and below 1). As the distribution becomes sufficiently close to the uniform distribution on [0, 1], this benefit approaches $\delta \cdot \big(1 - F(d^*)\big) \cdot (1 - d^*)$, which is strictly positive. Therefore, for any fixed $d^* < 1$, when F becomes sufficiently close to the uniform distribution on [0, 1], the benefit of cooperation for type d^* exceeds the cost, and hence the positive escalation continuous. Consequently, when F is everywhere close to the uniform distribution, the first period threshold d^* approaches 1 as the proportion of dominant types shrinks to zero, and hence almost full cooperation is attained.

The above intuition applies to distributions F that are everywhere close to the uniform distribution (see Fig. 1a). If, instead, the distribution F is close to the uniform distribution only for high types, that is, (8) holds but only for $\widehat{x} > 0$ (see Fig. 1b), then $F_1(x) = R(x)$ has multiple solutions , and the intuition is more complicated. In this case, it can be verified that if both players start with a belief that their counterparts cooperate with probability 0 in the first period, the escalation of positive expectations while leads to a positive cooperation, it is bounded away from 1, as $B \downarrow 1$. The positive escalation continues until 1 only if players start with a belief that their counterparts cooperate with a probability at least $F_B(\widehat{x})$.

Remark 1 Analogous to our Proposition 4, Baliga and Sjöström (2004) show that when the multiplier condition is satisfied, there exists a cheap-talk equilibrium under which almost full cooperation can be attained when the proportion of dominant types is sufficiently small. For their result, unlike ours, no additional conditions on type distributions is required. This difference stems from the different ways players "signal" their types in the two models.

In Baliga and Sjöström (2004), to attain a positive cooperation, the equilibrium is necessarily "non-monotonic" in the sense that there exists $0 < d_1 < d_2 < 1 - \mu$ such that the normal types ($d \in [0, d_1]$) and very high types ($d \in [d_2, 1 - \mu]$) pool together by sending the "Dove" signal, and the fairly high types ($d \in (d_1, d_2)$) separate out by sending the "Hawk" signal. If both players say Dove, then the normal type cooperates, and the very high type defects. It is shown that if the share of dominant types shrinks to 0, the equilibrium share of the normal types approaches 1, and almost full cooperation is attained. In this construction, conditional on a Dove signal, the cumulative distribution function has a flat part on the interval of fairly high types. As long as the flat part crosses the diagonal line, the multiplier condition is violated, and positive cooperation can be supported. As μ approaches 0 and B approaches 1 from above, the probability of a dominant type approaches 0 and the flat part can be made arbitrarily short. This yields almost full cooperation. The non-monotonicity in cheap-talk is essential to attain a cooperative outcome. Baliga and Sjöström (2004) show that if, instead, the cheap-talk is monotonic, then full defection is the unique equilibrium outcome.

In our paper, in the absence of cheap-talk, players signal their non-dominant types through cooperative actions, which is costly for high type players. Consequently, the equilibrium strategy is necessarily monotonic in the sense that there is a threshold d^* below which all types cooperate and above which all types defect (see Proposition 2). As argued before, to support an equilibrium where the first period threshold d^* is high, for high type players (types close to d^* from below), his loss from being cooperative (close to $d^* - F(d^*)$) has to be sufficiently small to be compensated by the gain from the counterpart's positive second period reaction to the cooperative signal. This is guaranteed by the conditions that F is sufficiently close to the uniform distribution on $[0, 1]$.

Remark 2 While both small μ and small B guarantee that the share of dominant types $1 - F(1 - \mu)$ is small, the value of μ plays an additional role. It is also the size of a player's additional satisfaction when the other player switches from defecting to cooperating, given that the player himself defects. In Proposition 3, to support s^* as an equilibrium, given the share of dominant types, the value μ has to be sufficiently small.[6] In s^* after the history (C, D), Player 1 who cooperates in period 1 knows that a non-dominant type of Player 2 will cooperate in the second period. If the proportion of dominant types is small, and in addition, μ is small, Player 1 too is better off cooperating again. However, given the same proportion of dominant types, if μ is sufficiently large, given the history (C, D), knowing that with high probability Player 2 will cooperate in the second period, Player 1 has a strong incentive to defect, in which case he enjoys both his outside option and a significant satisfaction from the cooperation attempt of Player 2. This incentive, if strong, can break the equilibrium s^*.

Remark 3 The role of δ in supporting an equilibrium with cooperation is not straightforward. On one hand, if μ can be made arbitrarily small, in particular if $\mu = 0$, then a larger δ implies a less restrictive condition (8),[7] which further implies a larger class of distribution functions that support an almost-full cooperation. However, if $\mu > 0$ is fixed, then a larger δ may hurt cooperation.

Note first that the first period threshold d^* of s^* is increasing in δ. Indeed, by cooperating rather than defecting in period 1, a player i of a relatively high type sacrifices the period 1 payoff to induce player j to cooperate in the second period. When δ increases, the benefit from the second period payoff accounts for a larger weight and hence a player with a higher outside option is willing to cooperate in the first period. When $\mu = 0$, high type players who cooperate in the first period are better off continue cooperating in the second period, and hence an equilibrium of the form s^* can be supported. However, when $\mu > 0$ is fixed, following the defection of player j in the first period, cooperating in the second period may no longer be beneficial for high types of i. As an example, suppose $F(x) = \frac{x}{1.02}$, and $\mu = 0.44$. It can be verified that s^* described in (4)–(5) is an equilibrium if $\delta = 0.7$, but it is not an equilibrium if $\delta = 0.9$.

We next illustrate our results with uniform distributions and with $\mu = 0$. Under the uniform distribution, condition (8) is satisfied for all $\delta \in (0, 1)$.[8] That is, as long

[6] Consider the following two games, where $\delta = 0.9$. In the first game $F_1(x) = \frac{x}{1.2}$ and $\mu_1 = 0.35$, and in the second game $F_2(x) = \frac{x}{1.02}$ and $\mu_2 = 0.4475$. It can be verified that the share of dominant types in the two games are the same $1 - F_1(1 - \mu_1) = 1 - F_2(1 - \mu_2)$, while s^* described in (4)–(5) is an equilibrium in the first game, but not in the second one.

[7] The uniform distribution on $[0, 1]$ is an exception since condition (8) holds for all $\delta \in (0, 1)$ (see Sect. 3).

[8] In fact, this is true not only for uniform distribution, but also for any distribution $F_1 \in \mathcal{G}_1$ that coincides with the uniform distribution on $[\widehat{x}, 1]$, for some $\widehat{x} \in (0, 1)$.

Table 2 The payoff structure for $\mu = 0$

1	2	
	D	C
D	d_1, d_2	$d_1, 0$
C	$0, d_2$	$1, 1$

as there is a positive (even if arbitrarily small) probability for a second interaction, almost full cooperation is attained when B is sufficiently close to 1 (from above).

3 Example: uniform distribution and $\mu = 0$

In this section, we focus on the uniform distribution $F_B^u = \frac{x}{B}$ on $[0, B]$ and assume $\mu = 0$. In addition to illustrating previous results, we show that except the full defection equilibrium, *all* symmetric PBEs (one of them is s^*) satisfy the property that almost-full cooperation is attained as $B \downarrow 1$.

Consider the game $G_1(F_B^u) := G_1(F_B^u, \mu = 0)$, where $B > 1$. The payoff structure is given in Table 2, which is Table 1 for $\mu = 0$.

By Corollary 1, full defection is the unique equilibrium in $G_1(F_B^u)$. We next turn to the twice repeated game $G_2(F_B^u) := G_2(F_B^u, \mu = 0)$. By the proof of Proposition 3, the following strategy profile s^* is an equilibrium. Since players are symmetric, we describe only Player 1's strategy. Let d_B^u be the unique solution[9] in $(0, 1)$ to (6), and let d_B^u and m_B^u be defined as in (7). Then $m_B^u > d_B^u$.

- In the first period, Player 1 of type d_1 cooperates iff $d_1 \leq d_B^u$.
- As for the second-period action, if the first-period action is
  - (C, C), Player 1 cooperates iff $d_1 \leq 1$.
  - (C, D), Player 1 cooperates iff $d_1 \leq m_B^u$.
  - (D, C), Player 1 cooperates iff $d_1 \leq 1$.
  - (D, D), Player 1 defects irrespective of his type.

Since $d_B^u < \min(m_B^u, 1)$, if player i follows s_i^* and chooses C in the first period, he must be of type $d_i \leq d_B^u$ and he continues to cooperate in the second period, regardless of the other player's first-period action.

Note that d_B^u is increasing to 1 as $B \downarrow 1$, regardless of δ. Interpreting δ as the probability of having a second interaction, regardless of how small this probability is, almost-full cooperation is achieved in both periods if the proportion of dominant types is sufficiently small. Figure 2 illustrates the increase in cooperation level in the two period game when δ is small ($\delta = 0.1$).

The above conclusion applies to the specific equilibrium s^* in $G_2(F_B^u)$. The next proposition shows that the same is true for *all* symmetric equilibria in $G_2(F_B^u)$, except for the full-defection equilibrium.

[9] It can be easily verified that $d_B^u = \frac{\delta B + B + \delta - 1 - \sqrt{(\delta+1)[(B+3)\delta + B - 1](B-1)}}{2\delta}$.

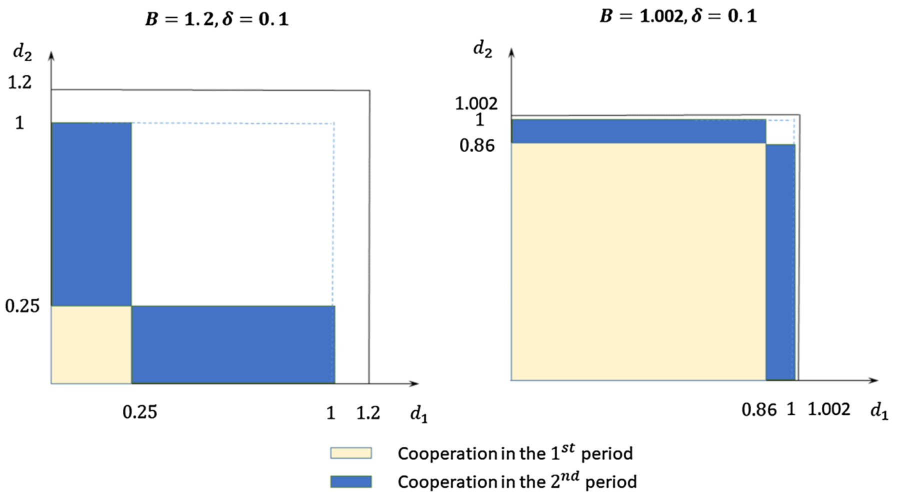


Fig. 2 Cooperation in $G_2(F_B^u)$ ($\delta = 0.1$)

Proposition 5 *Let $\delta \in (0, 1)$. For every $\epsilon > 0$, there exists $\eta > 0$ such that if $B \in (1, 1 + \eta)$, (i) full defection is the unique equilibrium of $G_1(F_B^u)$, and (ii) in every symmetric equilibrium of $G_2(F_B^u)$ other than the full-defection equilibrium, the players' first-period cooperation threshold $d^{\emptyset} > 1 - \epsilon$.*

Proof See Sect. A.5 of the "Appendix".

As shown in Lemma 3 in "Appendix A.5", for $B \in (1, 1 + \delta)$, except for the full-defection equilibrium, the symmetric PBE of $G_2(F_B^u)$ must be one of the following three forms. Since players are symmetric, we only provide the strategies for Player 1.

(a) $d_1^{\emptyset} > 0$, $d_1^{C,C} = 1$, and $d_1^{X,Y} = 0$ for all $(X, Y) \neq (C, C)$.
(b) $d_1^{\emptyset} > 0$, $d_1^{C,C} = 1$, $d_1^{C,D} = \frac{(1+\delta-B)d^{\emptyset}}{\delta}$, $d_1^{D,C} = \frac{1+\delta-B}{\delta}$, $d_i^{D,D} = 0$.
(c) $d_1^{\emptyset} > 0$, $d_1^{C,C} = 1$, $d_1^{C,D} = m^* > d^*$, $d_1^{D,C} = 1$ and $d_1^{D,D} = 0$. This is s^* defined in (4).

In these three equilibrium points, each player cooperates in the first period with a positive probability, and if both players cooperate (resp. defect) in period 1, they continue to cooperate (resp. defect) in period 2. The only difference lies on a players' action following a miscoordination in period 1. If in the first period Player 1 cooperates while his counterpart defects, then in the second period, Player 1 defects in equilibrium (a); he defects only with some probability in equilibrium (b); and he continues to cooperate in equilibrium (c).

Remark 4 For the uniform distribution, as shown in Proposition 5, as $B \downarrow 1$, all three symmetric equilibria (a)–(c) achieve cooperation with probability close to 1 ($d_1^{\emptyset} \uparrow 1$) and already in the first period. This, however, is not true for general distribution functions.

Example: Suppose $\delta = 0.9$, $F_1 = \frac{\sqrt{x}}{2-x}$, and $F_B(x) = F_1(\frac{x}{B})$ for $B > 1$. It can be verified that when players' type distributions follow F_B, both types of equilibrium points (a) and (c) exist. For equilibrium (a), as $B \downarrow 1$, the first period threshold $d_1^{\emptyset}$ approaches 1, but for equilibrium (c), $d_1^{\emptyset}$ is bounded away from 1.[10]

In all three types of equilibrium (a)–(c), a player's first period cooperation induces (in one way or another) the other player to cooperate in the second period. The escalation of positive expectations induces more types of players to cooperate already in the first period. To guarantee that the positive escalation continues until 1 as $B \downarrow 1$, it is further required that the players' type distribution functions to be sufficiently close to the uniform distribution. The intuition for this additional requirement is provided in Sect. 2.2 for equilibrium of type (c), and it also applies to other types of equilibrium. Nevertheless, since different types of equilibrium induce cooperation in different manners, the exact condition varies. In the above example, the distribution function $F_1 = \frac{\sqrt{x}}{2-x}$ satisfies the corresponding requirement for type (a) equilibrium (so that the first period threshold for this type of equilibrium approaches 1 as $B \downarrow 1$), but it violates that of type (c) equilibrium (so that the first period threshold for this type of equilibrium is bounded away from 1).

4 The T-period game with $\mu = 0$

In this section we extend the two-period game to any $T \geq 2$ period game, but for simplicity only under the assumption that $\mu = 0$. We show that Proposition 4 remains essentially the same when players interact for more than two periods, and, not surprisingly, it holds for a less restrictive class of distribution functions.

Suppose the distribution of players' types is given by $F_B : [0, B] \to [0, 1]$, where $B > 1$. Let $T \geq 3$. In the game $G_T(F_B) := G_T(F_B, \mu = 0)$, the players play the game $G_1(F_B, \mu = 0)$ (see Table 2) T times. The types of players do not change across periods. The payoff of a player is the discounted sum of his per period expected payoffs. Players of all types have the same discount factor $\delta \in (0, 1)$. The actions of the players are publicly observed after each period.

Consider the following strategy profile s_T^* of $G_T(F_B)$, where the players in the first two periods act as in s^* of $G_2(F_B)$ (given by (4) and (5)). Namely,

- In periods $t = 1, 2$:

$$s_{T1}^* = (d_1^{\emptyset} = d_T^*, d_1^{C,D} = m_T^*, d_1^{C,C} = d_1^{D,C} = 1 \text{ and } d_1^{D,D} = 0), \tag{9}$$

$$s_{T2}^* = (d_2^{\emptyset} = d_T^*, d_2^{D,C} = m_T^*, d_2^{C,C} = d_2^{C,D} = 1 \text{ and } d_2^{D,D} = 0), \tag{10}$$

  where $m_T^* > d_T^*$ are given in (12) and (13) below.
- If $T \geq 3$, in periods $3 \leq t \leq T$:

[10] Note that F_1 violates condition (8) of Proposition 4. It is left open the question whether the conditions of Proposition 4 guarantee that except for the full defection equilibrium, in every PBE (and not only the equilibrium s^* described in (4)–(5)) almost full cooperation is achieved in both periods, as $B \downarrow 1$.

- If the second-period pair of actions is (C, C), players of type $d \leq 1$ cooperate in every period t, and players with type $d > 1$ defect in every period t, $3 \leq t \leq T$.
- Otherwise, the players defect in every period t, $3 \leq t \leq T$, irrespective of their types.

In s_T^*, players with types $d \leq d_T^*$ cooperate in the first period. In the second period, a player of type $d \leq d_T^*$ continues to cooperate, regardless of the other player's first-period action; and a player with type $d \in (d_T^*, 1]$ cooperates if and only if the other player cooperates in the first period. For $T \geq 3$, and for every t, $3 \leq t \leq T$, if both players cooperate in the second period, they continue to cooperate in period t. Otherwise, both players defect in period t. Note that in s_T^*, if player i cooperates in the first two periods while player j defects in these periods, then i knows that j has a dominant type ($d_j > 1$) and i is best off choosing D.

Let b_T^* be the belief system that is computed from s_T^* using Bayes' rule wherever possible, and at each information set of player i that is reached with probability zero, player i believes that player j is of type $d_j > 1$ (a dominant-type). In "Appendix A.6" we show that the strategy profile s_T^* with the belief system b_T^* constitute a PBE of $G_T(F_B)$. Let

$$\Delta := \begin{cases} \sum_{t=2}^{T-1} \delta^t & \text{if } T \geq 3 \\ 0 & \text{if } T = 2. \end{cases} \tag{11}$$

The first period threshold d_T^* is the maximal solution in $(0, 1)$ to

$$d_T^* = \frac{F_B(d_T^*) + \delta \cdot (F_B(1) - F_B(d_T^*)) + \Delta \cdot (F_B(1) - F_B(d_T^*))}{1 + \delta(1 - F_B(d_T^*)) + \Delta \cdot (F_B(1) - F_B(d_T^*))}, \tag{12}$$

and m_T^* is

$$m_T^* := \frac{\delta \cdot (F_B(1) - F_B(d_T^*)) + \Delta \cdot (F_B(1) - F_B(d_T^*))}{\delta(1 - F_B(d_T^*)) + \Delta \cdot (F_B(1) - F_B(d_T^*))}. \tag{13}$$

Next, similar to Proposition 4, we provide a family of distributions on $[0, B]$, $B > 1$, for which full defection is the unique equilibrium of $G_1(F_B)$, almost-full cooperation is attained by s_T^* in $G_T(F_B)$, in all periods. That is, d_T^* is increasing to 1, as $B \downarrow 1$.

Proposition 6 *Let $T \geq 2$ and $\delta \in (0, 1)$. Suppose $F_1 \in \mathcal{G}_1$, and*

$$\text{there exists } \widehat{x} \in (0,1) \text{ s.t. } F_1(x) > \overbrace{\frac{x - (\delta + \Delta)(1 - x)}{1 - (\delta + \Delta)(1 - x)}}^{R_T(x)} \text{ for all } x \in (\widehat{x}, 1). \tag{14}$$

Then for every $\epsilon > 0$, there exists $\eta > 0$ such that for every $B \in (1, 1 + \eta)$, and $F_B \in \mathscr{F}(F_1)$,

(i) The full defection equilibrium is the unique equilibrium of the one-shot game $G_1(F_B, \mu = 0)$.
(ii) The T-period game $G_T(F_B, \mu = 0)$ has a PBE s_T^* defined in (12)–(13) where $d_T^* \in (1 - \epsilon, 1)$.
(iii) $R_T(x)$ is decreasing in T for every $x \in (\widehat{x}, 1)$.

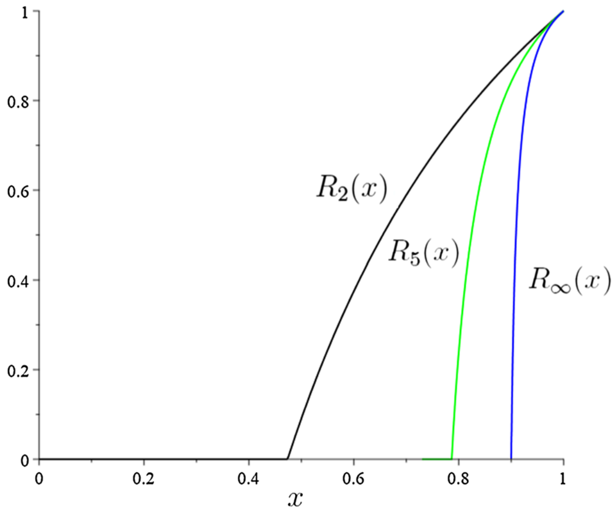


Fig. 3 The functions $R_2(x)$, $R_5(x)$, and $R_\infty(x)$ for $\delta = 0.9$

Proof See Sect. A.6 of the "Appendix". □

Proposition 6 states that, similar to Proposition 4, for every F_1 in a restrictive class of distributions on $[0, 1]$ and for $F_B \in \mathscr{F}(F_1)$, $B > 1$, the T-period game $G_T(F_B)$, $T \geq 2$, has a PBE where almost-full cooperation can be attained in each one of the T periods, if B approaches 1. In contrast, full defection is the unique equilibrium of the one-shot game $G_1(F_B)$.

Note that condition (14) is less restrictive than (8) since $R_T(x)$ is decreasing in T. Hence the sufficient condition (14) allows for larger class of distribution functions as the number of interactions increases. For $\delta = 0.9$, the functions $R_2(x)$, $R_5(x)$, and $R_\infty(x)$ are shown in Fig. 3.

5 Conclusion and future work

This paper studies a Stag Hunt game where players' payoffs from non-cooperation are their private information. For a class of distribution functions, which includes the uniform distribution, if there is only a small but positive probability that each player has a dominant type (that is, he is better off not cooperating, regardless of the action of the other player), non-cooperation by all types is the only equilibrium of the one-shot game. It is shown that this disappointing outcome may drastically change if there is a positive probability that the two players meet once more. In this case, if the proportion of dominant types is sufficiently small, almost-full cooperation is achieved in both periods, even if the probability of meeting for a second time is very small.

One corollary of this paper is that there are simple circumstances where limiting players to only one interaction results in a failure to cooperate, even though cooperation is more likely preferred by both of them. Leaving the door open for just one more interaction (even if its likelihood is small) may drastically change the outcome: with high probability the two players will cooperate in both periods.

In this paper, we assume that the players have identical distributions over types. One possible extension is to non-symmetric type distributions. Suppose that irrespective of his type, player i believes that the types of the other player j are distributed according to a distribution function $F_j : [0, B_j] \to [0, 1]$. An example is an R&D cooperation between two competing firms: one "large" and the other one "small". By a "large firm" we refer to a firm that is more likely to have a better outside option. It can be shown that if $\max(B_1, B_2) > 1 - \mu$, then the strategy profile $s^* = (s_1^*, s_2^*)$ is an equilibrium of the one-shot game if and only if $s_i^* = s_i^{\widehat{d}_i}$, $i = 1, 2$, where

$$(\widehat{d}_1, \widehat{d}_2) \in \{(d_1, d_2) | d_1 = (1 - \mu)F_2(d_2) \text{ and } d_2 = (1 - \mu)F_1(d_1)\}.$$

The analysis of the multi-period game with non-symmetric players is challenging.

Another extension is to the case where $\mu < 0$ (see Table 1). The case $\mu > 0$ fits scenarios where two enemies are engaged in arms race, while the case $\mu < 0$ is relevant to two friends rather than two enemies that are engaged in a joint project. Each feels guilty for not cooperating if his friend chooses to cooperate. In this case, suppose the players' type distributions follow the uniform distribution on $[0, B]$. If B is sufficiently large ($B > 1 + |\mu|$), then despite the players' inclination to cooperate, full defection is the unique equilibrium in the one-shot game. While in the two period game, the cooperative strategy s^* described in (4)–(5) remains an equilibrium if $|\mu|$ is sufficiently small. Moreover, the first period threshold of s^* approaches 1 as $B \downarrow 1$ and $\mu \downarrow 0$. The characterization of the set of distribution functions (other than the uniform one) for which this result holds, remains open.

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A Appendix

A.1 Proof of Proposition 1

For any pair of pure strategies (s_1, s_2), let

$$T_i = T_i(s_i) = \{d_i \in [0, B] | s_i(d_i) = C\}.$$

That is, T_i is the (Borel) set of all types of player i who cooperate under s_i. Since C is a weakly dominant strategy for i of type $d_i = 0$ it is assumed that $0 \in T_i$, and hence $T_i \neq \emptyset$. Let λ be the measure on $[0, B]$ generated by the distribution function F. The measure of T_i is therefore $\lambda(T_i)$.

If player i of type $d_i \leq 1 - \mu$ chooses C, his expected payoff is $\lambda(T_j)$. If he chooses D he obtains $d_i\big(1 - \lambda(T_j)\big) + (d_i + \mu)\lambda(T_j) = d_i + \mu\lambda(T_j)$. Hence $d_i \in T_i$ implies

$d_i \leq (1-\mu)\lambda(T_j)$, $j \neq i$, and $d_i \notin T_i$ implies $d_i \geq (1-\mu)\lambda(T_j)$. Consequently, in equilibrium, both T_1 and T_2 are intervals starting from zero. Part (i) of Proposition 1 follows. We next turn to proof part (ii).

Suppose $B > 1-\mu$. Let $s^* = (s_1^*, s_2^*)$ be an equilibrium of $G_1(F,\mu)$. By part (i), both s_1^* and s_2^* are threshold strategies. For $i = 1, 2$, denote by $\widehat{d_i}$ the threshold in s_i^*. As argued above, player i chooses C only if $d_i \leq (1-\mu)\lambda(T_j)$. Therefore, $\widehat{d_i} = \min[B, (1-\mu)\lambda(T_j)] = \min[B, (1-\mu)F(\widehat{d_j})]$. Since $B > 1-\mu$, we have for all $i = 1, 2$ and $j \neq i$,

$$\widehat{d_i} = (1-\mu)F(\widehat{d_j}). \tag{15}$$

Suppose w.l.o.g. $\widehat{d_i} \geq \widehat{d_j}$. Then

$$\widehat{d_i} = (1-\mu)F(\widehat{d_j}) \leq (1-\mu)F(\widehat{d_i}) = \widehat{d_j},$$

and hence $\widehat{d_i} = \widehat{d_j}$. Therefore, when s^* is an equilibrium, the threshold in s_i^* satisfies $\widehat{d_i} = (1-\mu)F(\widehat{d_i})$.

We next verify that as long as $\widehat{d}$ is a fixed point of F, the strategy profile $(s_1^{\widehat{d}}, s_2^{\widehat{d}})$ is an equilibrium of $G_1(F,\mu)$.

Suppose Player 1 plays $s_1^{\widehat{d}}$. Let us show that $s_2^{\widehat{d}}$ is best reply to $s_1^{\widehat{d}}$. If d_2 chooses C he obtains $F(\widehat{d})$, otherwise, he obtains $d_2 + \mu F(\widehat{d})$. Hence d_2 prefers C iff $d_2 \leq (1-\mu)F(\widehat{d}) = \widehat{d}$. This completes the proof of part (ii) in Proposition 1.

A.2 Proof of Proposition 2

Similar to Proposition 1, it can be verified that the second-period choice of each player is determined by thresholds. Namely, $s_i^*(d_i, h^1)$, $h^1 \in H^1$ is defined by (3) for some threshold $d_i^{h^1}$. We need to prove that this is also true for the first-period choice, namely, (3) also holds for $h = \emptyset$.

Let $X \in \{C, D\}$. Denote

$$\begin{aligned}
E_2^{\emptyset} &= \{d_2 \in [0,B] | s_2^*(d_2, \emptyset) = C\},\\
\overline{E}_2^{\emptyset} &= [0,B] \setminus E_2^{\emptyset},\\
E_2^{C,X} &= [0, d_2^{C,X}] \text{ and } E_2^{D,X} = [0, d_2^{D,X}],\\
P_2^{\emptyset} &= Prob E_2^{\emptyset}.
\end{aligned}$$

Suppose Player 1 of type d_1 chooses C in the first period. He obtains an expected payoff of $U_1(d_1, C)$, where

$$\begin{aligned}
U_1(d_1, C) = P_2^{\emptyset} + \delta P_2^{\emptyset} &\begin{cases} Prob(E_2^{C,C} | E_2^{\emptyset}) & d_1 \leq d_1^{C,C} \\ d_1 + \mu \cdot Prob(E_2^{C,C} | E_2^{\emptyset}) & d_1 > d_1^{C,C} \end{cases} +\\
&+ (1 - P_2^{\emptyset}) \cdot 0 + \delta(1 - P_2^{\emptyset}) \begin{cases} Prob(E_2^{C,D} | \overline{E}_2^{\emptyset}) & d_1 \leq d_1^{C,D} \\ d_1 + \mu \cdot Prob(E_2^{C,D} | \overline{E}_2^{\emptyset}) & d_1 > d_1^{C,D}. \end{cases}
\end{aligned}$$

Observe that if Player 2 chooses C in the first period then the updated probability distribution of Player 1 about Player 2's type changes from $F_2(d_2)$ to $F_2(d_2|d_2 \in E_2^{\emptyset})$. If Player 2 chooses D in the first period the updated distribution of Player 1 is $F_2(d_2|d_2 \in \overline{E}_2^{\emptyset})$. Since $B > 1-\mu$, the second-period thresholds satisfy $d_1^{h^1} < B$ for all $h^1 \in H^1$. By Proposition 1 for $X \in \{C, D\}$,

$$(1-\mu) \cdot Prob(E_2^{X,C}|E_2^{\emptyset}) = d_1^{X,C} \tag{16}$$

and

$$(1-\mu) \cdot Prob(E_2^{X,D}|\overline{E}_2^{\emptyset}) = d_1^{X,D}. \tag{17}$$

By (16) and (17)

$$\begin{aligned} U_1(d_1, C) = P_2^{\emptyset} + \delta P_2^{\emptyset} &\begin{cases} \frac{1}{1-\mu} \cdot d_1^{C,C} & d_1 \leq d_1^{C,C} \\ d_1 + \frac{\mu}{1-\mu} \cdot d_1^{C,C} & d_1 > d_1^{C,C} \end{cases} \\ + \delta(1-P_2^{\emptyset}) &\begin{cases} \frac{1}{1-\mu} \cdot d_1^{C,D} & d_1 \leq d_1^{C,D} \\ d_1 + \frac{\mu}{1-\mu} \cdot d_1^{C,D} & d_1 > d_1^{C,D}. \end{cases} \end{aligned} \tag{18}$$

By (18), $U_1(d_1, C)$ is continuous in d_1 (as a sum of two continuous functions) and it is piecewise linear in d_1. Moreover, the slope of $U_1(d_1, C)$ with respect to d_1 is at most δ.

Similarly, if Player 1 chooses D in the first period, he obtains

$$\begin{aligned} U_1(d_1, D) = p_2^{\emptyset} \cdot (d_1 + \mu) + \delta P_2^{\emptyset} &\begin{cases} Prob(E_2^{D,C}|E_2^{\emptyset}) & d_1 \leq d_1^{D,C} \\ d_1 + \mu \cdot Prob(E_2^{D,C}|E_2^{\emptyset}) & d_1 > d_1^{D,C} \end{cases} \\ + (1-p_2^{\emptyset}) \cdot d_1 + \delta(1-P_2^{\emptyset}) &\begin{cases} Prob(E_2^{D,D}|\overline{E}_2^{\emptyset}) & d_1 \leq d_1^{D,D} \\ d_1 + \mu \cdot Prob(E_2^{D,D}|\overline{E}_2^{\emptyset}) & d_1 > d_1^{D,D}. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} U_1(d_1, D) = p_2^{\emptyset} \cdot (d_1 + \mu) + \delta P_2^{\emptyset} &\begin{cases} \frac{1}{1-\mu} \cdot d_1^{D,C} & d_1 \leq d_1^{D,C} \\ d_1 + \frac{\mu}{1-\mu} \cdot d_1^{D,C} & d_1 > d_1^{D,C} \end{cases} \\ + (1-p_2^{\emptyset}) \cdot d_1 + \delta(1-P_2^{\emptyset}) &\begin{cases} \frac{1}{1-\mu} \cdot d_1^{D,D} & d_1 \leq d_1^{D,D} \\ d_1 + \frac{\mu}{1-\mu} d_1^{D,D} & d_1 > d_1^{D,D}. \end{cases} \end{aligned} \tag{19}$$

The payoff $U_1(d_1, D)$ is also continuous and piecewise linear in d_1, and its slope is at least 1. Since $\delta < 1$, it is easy to verify that $U_1(d_1, C) - U_1(d_1, D)$ is continuous in d_1 and has a negative slope everywhere in $[0, B]$. Namely, it is decreasing everywhere in $[0, B]$ and if $U_1(d_1, C) \geq U_1(d_1, D)$ for some d_1, it certainly holds for any type below d_1. This implies that $s_1^*(d_1, \emptyset)$ is a threshold strategy and $d_1^{\emptyset}$ is the (unique) solution of $U_1(d, C) = U_1(d, D)$ in d, if a solution exists. If $U_1(d_1, C) < U_1(d_1, D)$ for all $d_1 \in (0, B]$, then $d_1^{\emptyset} = 0$ and if $U_1(d_1, C) > U_1(d_1, D)$ for all $d_1 \in (0, B]$ then $d_1^{\emptyset} = B$.

A.3 Proof of Proposition 3

Since the game is symmetric and the strategy s^* described in (4)–(5) is symmetric, to show that s^* is an equilibrium, it is sufficient to show that s_1^* is a best response to s_2^*. There are two key steps: (1) there exists some $d^* \in (0, 1-\mu)$ such that it is optimal for Player 1 to cooperate in the first period if and only if $d_1 \leq d^*$, and (2) after Player 1 cooperated and his counterpart defected in period 1, Player 1 of type $d_1 \leq d^*$ strictly prefers to continue cooperating.

By defecting rather than cooperating in period 1, Player 1 of type d_1 gains the value of his outside option d_1, but loses $1-\mu$ if his counterpart chooses to cooperate, which happens if his counterpart is of type $d_2 \leq d^*$. Therefore, by defecting rather than cooperating in period 1, the payoff of Player 1 of type d_1 changes by

$$d_1 - F(d^*) \cdot (1-\mu) \tag{20}$$

Suppose the counterpart has cooperated in period 1, then the counterpart will continue cooperating in period 2 and and Player 1 of any non-dominant type will cooperate, regardless of Player 1's period 1 action. So Player 1's period 2 payoff does not depend on his own period 1 action, if the counterpart cooperated in period 1, which happens when the counterpart has type $d_2 \leq d^*$.

Suppose the counterpart has defected in period 1, which happens if and only if the counterpart's type is above d^*. By defecting rather than cooperating in period 1 and then following the equilibrium strategy, two possible effects will change Player 1's period 2 payoff: (1) he will defect rather than cooperate in period 2 if his type is $d_1 \leq m^*$ where $m^* > d^*$, and (2) his counterpart of type $d_2 \in (d^*, 1-\mu)$ will defect rather than cooperate in period 2.

The own action change effect, effect (1), is present if and only if Player 1's type is $d_1 < m^*$. Recall that, conditional on defection by the counterpart in period 1, Player 1 believes that the counterpart's type is above d^*. Suppose the counterpart acts as if Player 1 cooperated in period 1. Then, in period 2, the counterpart of type $d_2 \in (d^*, 1-\mu]$ will cooperate while the counterpart of type $d_2 > 1-\mu$ will defect. Given such behaviour from the counterpart, Player 1 of type d_1 gains his outside option value d_1, but loses $1-\mu$ if his counterpart cooperates, which happens with conditional probability $\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}$. Conditional on the counterpart defecting in period 1, effect (1) changes type d_1 Player 1's payoff by

$$d_1 - (1-\mu) \cdot \frac{F(1-\mu) - F(d^*)}{1 - F(d^*)} \tag{21}$$

if $d_i < m^*$. Note that in the analysis of effect (1), we assume that the counterpart acts as if Player 1 cooperated in period 1. Therefore, the amount in (21) is exactly type d_1 Player 1's gain from defection at history (C, D).

We next turn to effect (2). By defecting rather than cooperating in period 1, Player 1 causes the counterpart of type $d_2 \in (d^*, 1-\mu)$ to defect rather than cooperate in period 2. Given that Player 1 defects in period 2, effect (2) causes Player 1's period 2 payoff to go down by μ with conditional probability $\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}$. So, by defecting rather than cooperating in period 1 and then following the equilibrium strategy, type d_1 Player 1's period 2 payoff changes by

$$\left(1-F(d^*)\right)\cdot\left[\left(\underbrace{d_1-(1-\mu)\cdot\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}}_{\substack{\text{effect of Player 1's own action change,}\\ \text{fixing the counterpart's action.}}}\right)\mathbb{1}_{d_1\le m^*}-\mu\cdot\underbrace{\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}}_{\substack{\text{effect on the counterpart's}\\ \text{action change.}}}\right]. \tag{22}$$

Since the cut-off at history (C, D) is $m^* > d^*$, type d^* player is indifferent between defecting and cooperating in period 1 and then following the equilibrium strategy if and only if

$$\begin{aligned}0 =& d^*-F(d^*)\cdot(1-\mu)\\ &+\delta\cdot\left(1-F(d^*)\right)\cdot\left[\left(\underbrace{d^*-(1-\mu)\cdot\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}}_{\substack{\text{Part M: type } d^* \text{ Player 1's gain}\\ \text{by defecting rather than cooperation at history } (C,D).}}\right)-\mu\cdot\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}\right].\end{aligned} \tag{23}$$

It then follows that, the cut-off in the first period, d^*, is a solution to (23), which can be simplified to:

$$d^*=\frac{(1-\delta-\mu)F(d^*)+\delta F(1-\mu)}{1+\delta-\delta F(d^*)}, \tag{24}$$

It can be verified that the RHS of (23) is strictly negative when $d^*=0$, and it is strictly positive when $d^*=1-\mu$. Since the RHS of (23) is continuous in d^*, equation (23) has a solution in $(0, 1-\mu)$, for any $\mu\in[0, 1)$. In case there are multiple solutions, we let d^* be the minimum one in $(0, 1-\mu)$.

We now turn to prove that after Player 1 cooperated and his counterpart defected in period 1, Player 1 of type d^* strictly prefers to continue cooperating. As shown in (23), by defecting rather than cooperating at history (C, D), by (6) the conditional payoff of Player 1 of type d^* changes by

$$\begin{aligned}&d^*-(1-\mu)\cdot\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}\\ &=\frac{-\frac{1}{\delta}\cdot\left(d^*-F(d^*)\right)+\left(F(1-\mu)-F(d^*)\right)-\frac{\mu}{\delta}\cdot F(d^*)}{1-F(d^*)}\\ &-(1-\mu)\cdot\frac{F(1-\mu)-F(d^*)}{1-F(d^*)}\\ &=\frac{1}{1-F(d^*)}\cdot\left\{-\frac{d^*-F(d^*)}{\delta}+\mu\cdot\left[F(1-\mu)-\left(1+\frac{1}{\delta}\right)\cdot F(d^*)\right]\right\}.\end{aligned} \tag{25}$$

Since $d > F(d)$ for all $d\in(0, 1]$ and since $d^*\in(0, 1]$, (25) is strictly negative for the case $\mu=0$. That is, for $\mu=0$, type d^* of Player 1 is better off cooperating at history (C, D). Since following history (C, D), Player 1's second period strategy is of the threshold form, the corresponding threshold, denoted m^*, then satisfies $m^* > d^*$ for the case $\mu = 0$. The next lemma asserts that d^* as a function of μ is continuous at $\mu = 0$, and hence by continuity, formula (25) is negative for sufficiently small $\mu > 0$, as desired.

Lemma 1 *There exists $\tilde{\mu} > 0$ and a function $d^* : [0, \tilde{\mu}) \to (0, 1)$ such that $d^*(\mu)$ is a solution to (24) for all $\mu \in [0, \tilde{\mu})$, and $d^*(\mu)$ is continuous at $\mu = 0$. Furthermore, for all $\mu \in [0, \tilde{\mu})$, $d^*(\mu) < 1 - \mu$.*

Proof Let

$$H(\mu, d) := \frac{(1 - \delta - \mu)F(d) + \delta F(1 - \mu)}{1 + \delta - \delta F(d)}. \tag{26}$$

Let $G : [0, 1] \times [0, 1] \to \mathbb{R}$ be

$$G(\mu, d) := H(\mu, d) - d. \tag{27}$$

For $\mu = 0$, let d_0 be the *minimal* solution to $G(0, d) = 0$ on $(0, 1)$. Here d_0 is well defined because (i) $G(0, d = 0) > 0$ and $G(0, d = 1) < 1$, and hence a solution to $G(0, d) = 0$ on $(0, 1)$ exists; (ii) the function G is continuous function on a compact set, hence the smallest solution exists.

Let $\mu > 0$. Since $G(0, d_0) = 0$ and $G(\mu, d)$ is decreasing in μ, $G(\mu, d_0) < 0$ for all $0 < \mu < 1$. Moreover, since $G(0, 0) > 0$, and $G(\mu, d)$ is continuous in μ, for sufficiently small $\mu > 0$, $G(\mu, 0) > 0$. Namely, there exists $\tilde{\mu}_1 > 0$ such that $G(\mu, 0) > 0$ for all $\mu \in (0, \tilde{\mu}_1]$. Therefore, for $\mu \in (0, \tilde{\mu}_1]$, there exists a solution $d \in (0, d_0]$ to $G(\mu, d) = 0$. Let $d^*(\mu)$ be the *maximal* solution in d to $G(\mu, d) = 0$ in $(0, d_0]$. Here, again, $d^*(\mu)$ is well defined since G is continuous.

We next prove the continuity of $d^*(\mu)$ at $\mu = 0$. We will show that for every sequence $(\mu_n)_{n\in\mathbb{N}}$ that converges to 0, the sequence $\left(d^*(\mu_n)\right)_{n\in\mathbb{N}}$ converges to d_0, where $d_0 := d^*(0)$.

Suppose to the contrary that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ that converges to 0, under which the sequence $\left(d^*(\mu_n)\right)_{n\in\mathbb{N}}$ does not converge to d_0. Then there exists $\epsilon > 0$ (without loss of generality we can assume that $\epsilon < 3d_0$) such that for every $N \in \mathbb{N}$, there exists $n \geq N$ with $|d^*(\mu_n) - d_0| \geq \epsilon$. Since $G(0, 0) > 0$ and d_0 is the minimal solution to $G(0, d) = 0$, we have $G(0, d) > 0$ for all $d \in [0, d_0)$. This, together with $\frac{\epsilon}{3} < d_0$, imply that $G(0, d_0 - \frac{\epsilon}{3}) > 0$. Hence, for N sufficiently large, and for all $n \geq N$, we have $G(\mu_n, d_0 - \frac{\epsilon}{3}) > 0$ and $G(\mu_n, d_0) < G(0, d_0) = 0$. Therefore, the equation $G(\mu_n, d) = 0$ has a solution in $[d_0 - \frac{\epsilon}{3}, d_0]$. Since $d^*(\mu_n)$ is defined as the largest solution in $[0, d_0]$ to $G(\mu_n, d) = 0$, we have $d^*(\mu_n) \in [d_0 - \frac{\epsilon}{3}, d_0]$, contradicting $|d^*(\mu_n) - d_0| \geq \epsilon$. Hence for every sequence $(\mu_n)_{n\in\mathbb{N}}$ that converges to 0, the sequence $\left(d^*(\mu_n)\right)_{n\in\mathbb{N}}$ converges to d_0, as claimed.

We next argue that for small μ, $d^*(\mu) < 1 - \mu$. Indeed, since $d^*(\mu) + \mu$ is continuous in μ at $\mu = 0$ and $d^*(0) + 0 < 1$, there exists $\tilde{\mu}_2 > 0$ such that for all $\mu \in [0, \tilde{\mu}_2)$, we have $d^*(\mu) + \mu < 1$. The proof of Lemma 1 is complete by letting $\tilde{\mu} = \min(\tilde{\mu}_1, \tilde{\mu}_2)$. □

We have thus shown that for sufficiently small μ, Player 1 of type d^* is indifferent between cooperating and defecting in period 1, and he is better off continuing cooperation after the history (C, D). By (21) (resp. the effect of Player 1's own action change in (22)), Player 1's payoff change if he defects rather than cooperates in period 1 (resp. after the history (C, D)) is strictly decreasing in his type d_1. Hence (i) it is optimal for Player 1 to cooperate in the first period if and only if $d_1 \leq d^*$, and (ii) following (C, D), Player 1 of type $d_1 \leq d^*$ strictly prefers to continue cooperating.

To complete the proof that s^* is an equilibrium, it is left to verify that Player 1 has no incentive to deviate from s_1^* after histories (D, D), (C, C), and (D, C). If both players defect in period 1, then Player 2 must be of a type above d^*, and thus he defects for sure in period 2, making it optimal for Player 1 to defect. If Player 2 cooperated in period 1, then he must be of type $d_2 \leq d^*$, and thus continues cooperating in period 2, making it optimal for Player 1 to cooperate.

A.4 Proof of Proposition 4

Suppose $F_1 \in \mathcal{G}_1$. By Definition 2, for every $B > 1$, $\mu \geq 0$, and $x \in (0, 1]$, we have $(1-\mu)F_B(x) \leq F_B(x) = F_1(\frac{x}{B}) < F_1(x) \leq x$. By Corollary 1, full defection is the unique equilibrium of $G_1(F_B, \mu)$.

Suppose μ is sufficiently small and it satisfies Proposition 3 (that is, $\mu \in [0, \widehat{\mu})$). Since $B > 1$ and $F_B(x) < x$ for all $x \in (0, 1]$, by Proposition 3, the strategy profile s^* defined in (4)–(5) is an equilibrium of $G_2(F_B, \mu)$.

We first analyze the case $\mu = 0$. Define

$$H_B(x) := \frac{(1-\delta)F_B(x) + \delta \cdot F_B(1)}{1 + \delta - \delta \cdot F_B(x)}. \tag{28}$$

By (24), the first-period threshold d^* of s^* in $G_2(F_B, \mu = 0)$ is a fixed point of H_B.

Since F_1 satisfies (8) and since

$$H_1(x) > x \text{ iff } F_1(x) > \frac{x - \delta(1-x)}{1 - \delta(1-x)}, \tag{29}$$

there exists $\widehat{x} \in (0, 1)$ such that $H_1(x) > x$ for all $x \in (\widehat{x}, 1)$. Let $\epsilon > 0$. There exists $x_\epsilon \in (1 - \frac{\epsilon}{2}, 1)$ such that $H_1(x_\epsilon) > x_\epsilon$. By the uniform continuity of $H_B(x)$ as a bivariate function of (B, x) on $[1, 2] \times [0, 1]$, there exists $B_\epsilon > 1$ such that for every $1 < B < B_\epsilon$, $H_B(x_\epsilon) > x_\epsilon$. Let $B \in (1, B_\epsilon)$. By (28), $H_B(1) < 1$. By the continuity of $H_B(x)$, the equation $H_B(d) = d$ has solutions in $(x_\epsilon, 1)$. Denote by d_0^* the minimal solution to $H_B(d) = d$ in $(x_\epsilon, 1)$.

Similar to the proof of Lemma 1, there exists $\tilde{\mu} > 0$ and a unique continuous function $d^*(\mu) : [0, \tilde{\mu}) \to (0, 1-\mu)$ such that $d^*(0) = d_0^*$ and $d^*(\mu)$ satisfies (6). Therefore, there exists μ', $0 < \mu' < \tilde{\mu}$, such that for all $\mu \in [0, \mu')$, $d^*(\mu) - d_0^* < \frac{\epsilon}{2}$. Since $d_0^* \in (x_\epsilon, 1)$ and $x_\epsilon \in (1 - \frac{\epsilon}{2}, 1)$, we have $d^*(\mu) \in (1 - \epsilon, 1 - \mu)$ for all $\mu \in [0, \mu')$. That is, the two-period game $G_2(F_B, \mu)$ has an equilibrium s^* defined in (4)–(5) where $d^* \in (1 - \epsilon, 1 - \mu)$, as claimed in Proposition 4.

A.5 Proof of Proposition 5

Let $F_B^u : [0, B] \to \mathbb{R}_+$ be the uniform distribution on $[0, B]$, $B > 1$. In this section we characterize all symmetric equilibrium of $G_2(F_B^u)$. We will use the following lemma, the proof of which is straightforward and hence omitted.

Lemma 2 *Let* $\mu = 0$, *and* $F(x) = \frac{x}{B}$, $x \in [0, B]$.

(i) Suppose $B < 1$. *Then* $G_1(F, \mu)$ *has exactly two equilibrium points:* (s_1^0, s_2^0) *and* (s_1^B, s_2^B).

(ii) Suppose $B = 1$; *then every* $y \in [0, 1]$ *is a fixed point of* $F(\cdot)$ *and the set of equilibria of* $G_1(F, \mu)$ *is* $\{(s_1^y, s_2^y) | y \in [0, 1]\}$.

(iii) Suppose $B > 1$. *The only equilibrium of* $G_1(F, \mu)$ *is* (s_1^0, s_2^0).

As shown in Proposition 2, all equilibria of $G_2(F_B^u)$ consist of threshold strategies. We will first show that given the second-period thresholds, the first-period threshold is uniquely determined. We then characterize the thresholds after any history. Then we go back to determine the first-period threshold. This procedure yields at most 4 different equilibrium points. The next lemma deals with all symmetric equilibria other than the fully defecting equilibrium.

Lemma 3 *Except for the full-defection equilibrium, the game* $G_2(F_B^u)$ *has at most three symmetric equilibria. They are defined by the following thresholds:*

(i) For $1 < B < 1 + \delta$, $\gamma_1 = (d^{\emptyset} = \frac{1+\delta-B}{\delta}, d_i^{C,C} = 1, d_i^{X,Y} = 0$ *for all* $(X, Y) \neq (C, C))$.

(ii) $\gamma_2 = (d^{\emptyset} = \frac{2\delta}{b+\sqrt{b^2-4\delta^2}}$ *where* $b = (1+\delta)(B-1)+2\delta$, $d_i^{C,C} = d_1^{D,C} = d_2^{C,D} = 1, d_1^{C,D} = d_2^{D,C} = \frac{1-d^{\emptyset}}{B-d^{\emptyset}}, d_i^{D,D} = 0)$.

(iii) For $1 < B < 1+\delta$, $\gamma_3 = \left(d^{\emptyset} = \frac{1}{2}\left[B + \frac{\delta}{1+\delta-B} - \sqrt{\left(B + \frac{\delta}{1+\delta-B}\right)^2 - 4}\right], d_i^{C,C} = 1, d_1^{C,D} = d_2^{D,C} = \frac{(1+\delta-B)d^{\emptyset}}{\delta}, d_1^{D,C} = d_2^{C,D} = \frac{1+\delta-B}{\delta}, d_i^{D,D} = 0\right)$.

Proof Let $s^* = (s_1^*, s_2^*)$ be an equilibrium of $G_2(F_B^u)$, where $F_B^u(x) = \frac{x}{B}$, $B > 1$

Claim 1 *Given the thresholds* $d_i^{h^1}$, $h^1 \in H^1$, $i = 1, 2$, *the first-period threshold,* $d_i^{\emptyset}$, *is uniquely determined.*

Proof The proof of Proposition 2 shows that if s^* is an equilibrium of $G_2(F_B^u)$, $\Delta(d_i) \equiv U_i(d_i, C) - U_i(d_i, D)$ is strictly decreasing in d_i. Therefore the threshold $d_i^{\emptyset}$ is the unique solution of $\Delta(d_i) = 0$. If $\Delta(d_i) < 0$ for all $d_i \in [0, 1]$, then $d_i^{\emptyset} = 0$ and $d_i^{\emptyset} = 1$ if $\Delta d_i > 0$ for all $d_i \in [0, 1]$. □

Denote $d^{\emptyset} = d^*$. We next characterize the thresholds $d_i^{h^1}$, $h^1 \in H^1$. Let us start with $h^1 = (C, C)$. The updated belief of Player 1 over the types of Player 2 is $\widehat{F}_{\widehat{B}}^u(d_2)$ on $[0, \widehat{B}]$, and

$$\widehat{F}_{\widehat{B}}^u(d_2) = F_B^u(d_2 | E_2^{\emptyset}), \quad \widehat{B} = d^*,$$

where $E_2^{\emptyset}$ is the set of all types of 2 that choose C in the first period. By Proposition 2, $E_2^{\emptyset} = [0, d^*]$. By Lemma 2,

$$d_1^{C,C} = \min\left(\widehat{F}_{\widehat{B}}^u(d_2^{C,C}), B\right) = \widehat{F}_{\widehat{B}}^u(d_2^{C,C}) \tag{30}$$

and

$$\widehat{F}_{\widehat{B}}^u(d_2^{C,C}) = F_B^u(d_2^{C,C} | d_2 \in [0, d^*]).$$

Thus

$$d_1^{C,C} = \begin{cases} \frac{d_2^{C,C}}{d^*} & \text{if } d_2^{C,C} \le d^* \\ 1 & \text{if } d_2^{C,C} \ge d^*. \end{cases} \tag{31}$$

Similarly,

$$d_2^{C,C} = \begin{cases} \frac{d_1^{C,C}}{d^*} & \text{if } d_1^{C,C} \le d^* \\ 1 & \text{if } d_1^{C,C} \ge d^*. \end{cases} \tag{32}$$

By (30), (31), and (32), either $d^* = 1$ and $0 \le d_1^{C,C} = d_2^{C,C} \le 1$ or $d^* < 1$ and either $d_i^{C,C} = 1$ or $d_i^{C,C} = 0$.

Suppose next $h^1 = (D, D)$. By Lemma 2,

$$d_1^{D,D} = F_B^u(d_2^{D,D} | d_2 \notin E_2^{\emptyset}) = \frac{Prob([0, d_2^{D,D}] \cap [d^*, B])}{Prob([d^*, B])}. \tag{33}$$

Hence

$$d_1^{D,D} = \begin{cases} 0 & \text{if } d_2^{D,D} \le d^* \\ \frac{d_2^{D,D} - d^*}{B - d^*} & \text{if } d_2^{D,D} \ge d^*. \end{cases} \tag{34}$$

Similarly,

$$d_2^{D,D} = \begin{cases} 0 & \text{if } d_1^{D,D} \le d^* \\ \frac{d_1^{D,D} - d^*}{B - d^*} & \text{if } d_1^{D,D} \ge d^*. \end{cases} \tag{35}$$

Suppose $d_2^{D,D} > d^*$. By (34), $d_1^{D,D} > 0$, and by (35), $d_1^{D,D} > d^*$. Hence (34) and (35) imply $d_1^{D,D} = d_2^{D,D} \equiv d$, where

$$d = \frac{d - d^*}{B - d^*}.$$

Equivalently $d = \frac{d^*}{1 + d^* - B}$. Since $B > 1$, either $d < 0$ or $d > 1$, a contradiction. We conclude that $d_i^{D,D} = 0$.

Suppose next $h^1 = (C, D)$. Then by Lemma 2,

$$d_1^{C,D} = F_B^u(d_2^{C,D} | d_2 \notin E_2^{\emptyset}) = \frac{Prob([0, d_2^{C,D}] \cap [d^*, B])}{Prob([d^*, B])}.$$

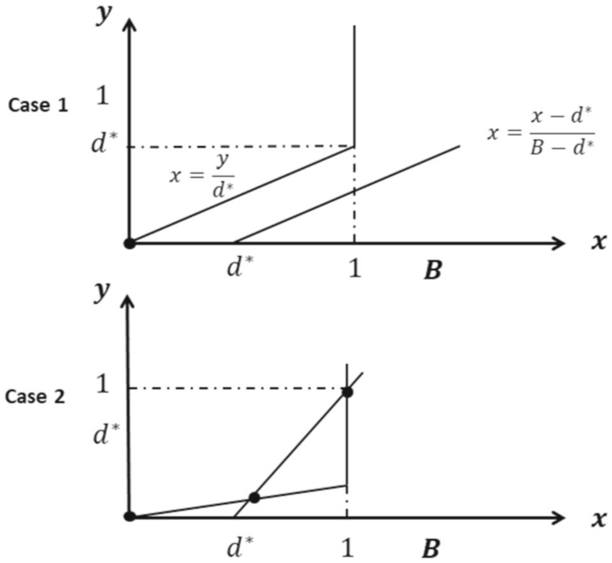

Fig. 4 Graphs of (38) and (39)

Consequently,

$$d_1^{C,D} = \begin{cases} 0 & \text{if } d_2^{C,D} \le d^* \\ \frac{d_2^{C,D} - d^*}{B - d^*} & \text{if } d_2^{C,D} \ge d^*. \end{cases} \tag{36}$$

Similarly,

$$d_1^{D,C} = \begin{cases} 1 & \text{if } d^* \le d_2^{D,C} \\ \frac{d_2^{D,C}}{d^*} & \text{if } d^* > d_2^{D,C}. \end{cases} \tag{37}$$

Let $d_1^{D,C} = d_2^{C,D} = x$ and $d_1^{C,D} = d_2^{D,C} = y$ (symmetric equilibrium). By (36) and (37),

$$y = \begin{cases} 0 & \text{if } x \le d^* \\ \frac{x - d^*}{B - d^*} & \text{if } x \ge d^*. \end{cases} \tag{38}$$

$$x = \begin{cases} 1 & \text{if } y \ge d^* \\ \frac{y}{d^*} & \text{if } y \le d^*. \end{cases} \tag{39}$$

There are two cases (see Fig. 4):

In Case 1 of Fig. 4 the only solution is $x = y = 0$. In Case 2 there are 3 solutions: (i) $(0, 0)$; (ii) $\left(1, \frac{1-d^*}{B-d^*}\right)$, where $\frac{1-d^*}{B-d^*} \in (0, 1)$ and (iii) $\left(\frac{d^*}{1-d^*(B-d^*)}, \frac{(d^*)^2}{1-d^*(B-d^*)}\right)$ provided $\frac{(d^*)^2}{1-d^*(B-d^*)} \le d^*$ and $\frac{d^*}{1-d^*(B-d^*)} \ge d^*$. It is easy to verify that the last two inequalities hold iff either $d^* = 0$ or $d^* \le \frac{B+1-\sqrt{(B+1)^2-4}}{2}$. Hence $0 \le d^* < 1$ must hold and either $d_i^{C,C} = 0$ or $d_i^{C,C} = 1$ (see the sentence below (32)). Let us examine these three cases.

Solutions (i) and (ii): Suppose $d_1^{X,Y} = d_2^{X,Y} = 0$ for $X \neq Y$, $X, Y \in \{C, D\}$, $d_i^{D,D} = 0$, and $d^* < 1$. Then either $d_i^{C,C} = 1$ or $d_i^{C,C} = 0$. Suppose first that $d_i^{C,C} = 1$. If Player 1 of type $d_1 \leq 1$ chooses C in period 1 he obtains

$$\begin{aligned} U_1(C, d_1) &= \frac{d^*}{B} + \delta\big[Prob(d_2 \leq d^*) \cdot 1 + Prob(d_2 > d^*) \cdot d_1\big] \\ &= (1+\delta)\frac{d^*}{B} + \frac{\delta(B - d^*)}{B} \cdot d_1. \end{aligned} \tag{40}$$

If 1 chooses D in period 1 he obtains $U_1(D, d_1) = (1+\delta)d_1$. Hence d^* is the solution to

$$(1+\delta)\frac{d^*}{B} + \frac{\delta(B - d^*)d^*}{B} = (1+\delta)d^*.$$

There are two solutions to the last equation. The first one is $d^* = \frac{1+\delta-B}{B}$ and for $B < 1+\delta$, $0 < d^* < 1$. In this case $d_i^{X,Y} = 0$ for any $(X, Y) \neq (C, C)$ and $d_i^{C,C} = 1$. The other solution is $d^* = 0$ with $d_i^{D,D} = 0$. On the equilibrium path every type of every player (except for type 0) defects in both periods. Since any history h^1 other than (D, D) is off the equilibrium path there are no restrictions on beliefs following h^1.

Suppose now $d_i^{C,C} = 0$. Similarly to (40), $U_1(C, d_1) = \frac{d^*}{B} + \frac{\delta(B-d^*)}{B} d_1$ and $U_1(D, d_1) = (1+\delta)d_1$. Since $U_1(C, d_1) \geq U_1(D, d_1)$ iff $d_1 \leq \frac{(1-\delta)d^*}{B}$ we must have $d^* = \frac{(1-\delta)d^*}{B}$ and again $d^* = 0$.

To complete the analysis of solution (i) we prove that $d^* \neq 1$. Suppose that $d^* = 1$ and $d_i^{C,C} \leq 1$ (see the sentence below (32)):

$$U_1(C, d_1) = \frac{d^*}{B} + \delta[Prob(d_2 \leq d^*)Prob(d_2 \leq d_2^{C,C} | d_2 \leq d^*) + Prob(d_2 > d^*)d_1],$$

$$U_1(C, d_1) = \frac{d^*}{B} + \frac{\delta d_2^{C,C}}{B} + \frac{\delta(B - d^*)d_1}{B},$$

while

$$U_1(D, d_1) = (1+\delta)d_1.$$

Since $d^* = 1$, we have

$$\frac{1}{B} + \frac{\delta d_2^{C,C}}{B} + \frac{\delta(B-1) \cdot 1}{B} = 1 + \delta.$$

Equivalently, $1 + \delta d_2^{C,C} - \delta = B$, but this contradicts $B > 1$.

The solution (ii) is essentially the strategy profile described in (4)–(5). Since it has been thoroughly studied in "Appendix A.3", we omit the detailed analysis here.

Finally, let us analyze solution (iii) of Case 2 in Fig. 4. Let

$$\gamma_3 = (d^{\emptyset} = d^*, d_i^{C,C} = 1, d_i^{D,D} = 0, d_1^{D,C} = d_2^{C,D} = x = \frac{d^*}{(d^*)^2 - Bd^* + 1},$$
$$d_1^{C,D} = d_2^{D,C} = y = \frac{(d^*)^2}{(d^*)^2 - Bd^* + 1}).$$

Note, that $0 < x \le 1$ iff

$$d^* \le \frac{B + 1 - \sqrt{(B+1)^2 - 4}}{2}. \tag{41}$$

In this case, $y = d^*x \le d^* < x$. The last inequality holds since $B > 1$. It can be easily verified that in this case (38) holds. Hence γ_3 defines an equilibrium iff (41) holds. Suppose that $d_1 \le 1$. Then,

$$\begin{aligned} U_1(C, d_1) &= (1+\delta)Prob(d_2 \le d^*) + \delta Prob(d_2 > d^*) \\ &\quad \cdot \begin{cases} Prob(d_2 \le x | d_2 > d^*) & d_1 \le y \\ d_1 & d_1 > y \end{cases} = \\ &= \frac{(1+\delta)d^*}{B} + \frac{\delta(B - d^*)}{B} \cdot \begin{cases} \frac{x - d^*}{B - d^*} & d_1 \le y \\ d_1 & d_1 > y. \end{cases} \end{aligned} \tag{42}$$

Next,

$$\begin{aligned} U_1(D, d_1) &= d_1 + \delta[Prob(d_2 \le d^*) \cdot \\ &\quad \begin{cases} Prob(d_2 \le y | d_2 \le d^*) & d_1 \le x \\ d_1 & d_1 > x \end{cases} + d_1 Prob(d_2 > d^*)] \\ &= d_1 + \delta \begin{cases} \frac{y}{B} & d_1 \le x \\ d_1 \frac{d^*}{B} & d_1 > x \end{cases} + \frac{\delta(B - d^*)d_1}{B}. \end{aligned} \tag{43}$$

<u>Subcase 1</u> Suppose $d_1 \le y$. Since $y \le x$, $U_1(C, d_1) \ge U_1(D, d_1)$ iff

$$\frac{(1+\delta)d^*}{B} + \frac{\delta(x - d^*)}{B} \ge d_1 + \frac{\delta y}{B} + \frac{\delta(B - d^*)d_1}{B}.$$

By the definition of γ_3, $y = d^*x$,

$$d_1[1 + \frac{\delta(B - d^*)}{B}] \le \frac{(1+\delta)d^*}{B} + \frac{\delta(x - d^*)}{B} - \frac{\delta d^* x}{B}$$

and

$$d_1 \le \frac{d^* + \delta(x - y)}{B(1+\delta) - \delta d^*} \equiv \widehat{d}.$$

Next,

$$\widehat{d} \geq y \text{ iff } \frac{d^* + \frac{\delta d^*(1-d^*)}{(d^*)^2 - Bd^* + 1}}{B(1+\delta) - \delta d^*} > \frac{(d^*)^2}{(d^*)^2 - Bd^* + 1}$$

or, equivalently,

$$\widehat{d} \geq y \text{ iff } (1+\delta)(d^*)^2 - \delta^*((2+\delta)B + \delta) + (1+\delta) \geq 0. \tag{44}$$

Note, that no equilibrium exists with $\widehat{d} < y$, since in equilibrium $y < d^*$ and $\widehat{d} = d^*$ must hold. So any candidate for an equilibrium of the next two subcases must have $\widehat{d} \geq y$.

Subcase 2 Suppose $y < d_1 \leq x$. Then

$$U_1(C, d_1) \geq U_1(D, d_1) \text{ iff } d_1 \leq \frac{(1+\delta)d^* - \delta y}{B} \equiv R_1.$$

Subcase 3 Suppose $d_1 > x$. Then $U_1(C, d_1) \geq U_1(D, d_1)$ iff

$$\frac{(1+\delta)d^*}{B} + \frac{\delta(B-d^*)d_1}{B} \geq d_1 + \frac{\delta d_1 d^*}{B} + \frac{\delta(B-d^*)d_1}{B}.$$

$$d_1 \leq \frac{(1+\delta)d^*}{B+\delta d^*} \equiv R_2 \tag{45}$$

It is easy to verify that $x \leq R_2$ iff $x \leq R_1$ and in this case $R_2 \leq R_1$. Consequently, either $x \leq R_2 \leq R_1$ or $x \geq R_2 \geq R_1$.

There are four cases to check (recall that $y \leq d^* < x$ must hold).
(1) $y < x \leq R_2 \leq R_1$. In this case, $d^* = R_2$ and $x \leq d^*$, a contradiction. (2) $R_1 \leq R_2 \leq y \leq x$. In this case, $d^* = y$ and $x = 1$. That is, $\frac{d^*}{(d^*)^2 - Bd^* + 1} = 1$ or $(d^*)^2 - Bd^* + 1 = d^*$. But it is easy to verify from (44) that in this case $\widehat{d} < y$ and it yields no equilibrium. (3) $R_1 \leq y \leq R_2 \leq x$. In this case again $d^* = y$. (4) $y \leq R_1 \leq R_2 < x$. In this case, $d^* = R_1 = \frac{(1+\delta)d^* - \delta y}{B}$. For $1 < B < 1+\delta$, $y = \frac{(1+\delta-B)d^*}{\delta} < d^*$ and $x = \frac{1+\delta-B}{\delta}$. It is easy to verify that (i) (41) holds, (ii) $x > R_1 \geq R_2 = d^*$, and (iii) (44) holds. Since

$$x = \frac{d^*}{(d^*)^2 - Bd^* + 1} = \frac{1+\delta-B}{\delta}$$

we have

$$d^* = \frac{2}{B + \frac{\delta}{1+\delta-B} + \sqrt{(B + \frac{\delta}{1+\delta-B})^2 - 4}},$$

where d^* is decreasing in B and $\lim_{B\downarrow 1} d^* \nearrow 1$. This completes the proof of Lemma 3.

A.6 Proof of Proposition 6

Suppose $\mu = 0$ and let $F_B \in \mathscr{F}(F_1)$, where $B > 1$. It is shown in "Appendix A.4" that full defection is the unique equilibrium of $G_1(F_B)$ if $F_1 \in \mathcal{G}_1$. We next deal with distributions in $\mathcal{G}_1$ and identify a subset of $\mathcal{G}_1$ for which $G_T(F_B)$ has an equilibrium where d_T^* defined in (12) converges to 1, as the proportion of the dominant types decreases to 0.

Step 1: For every $T \geq 3$, $d_T^* < m_T^*$.

By the Mean Value Theorem, it can be verified that equation (12) has a solution d_T^* in (0, 1). By (12),

$$d_T^* \cdot \delta \cdot \big(1 - F_B(d_T^*)\big) \\ = -\big(d_T^* - F_B(d_T^*)\big) + \delta \cdot \big(F_B(1) - F_B(d_T^*)\big) + (1 - d_T^*) \cdot \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big). \tag{46}$$

Since $F_1 \in \mathcal{G}_1$ and since the function F_B is generated by F_1 with $B > 1$, by the definitions of F_B and $\mathcal{G}_1$, we have $F_B(x) = F_1(\frac{x}{B}) < F_1(x) \leq x$. Hence

$$F_B(x) < x \quad \text{for all} \quad x \in (0, 1]. \tag{47}$$

This observation, together with $d_T^* \in (0, 1)$, imply that $d_T^* > F_B(d_T^*)$. By (46),

$$d_T^* \cdot \delta \cdot \big(1 - F_B(d_T^*)\big) < \delta \cdot \big(F_B(1) - F_B(d_T^*)\big) + (1 - d_T^*) \cdot \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big). \tag{48}$$

This implies

$$d_T^* < \frac{\delta \cdot \big(F_B(1) - F_B(d_T^*)\big)}{\delta \cdot \big(1 - F_B(d_T^*)\big)} + (1 - d_T^*) \cdot \frac{\Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)}{\delta \cdot \big(1 - F_B(d_T^*)\big)}. \tag{49}$$

Equivalently,

$$d_T^* < \frac{\delta \cdot \big(F_B(1) - F_B(d_T^*)\big) + \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)}{\delta \cdot \big(1 - F_B(d_T^*)\big) + \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)} \\ = m_T^*. \tag{50}$$

Step 2: In this step we show that the strategy profile $s_T^* = (s_{T1}^*, s_{T2}^*)$ defined in Sect. 4 (see (9)–(10) for the first two periods) is an equilibrium of $G_T(F_B)$.

Let us show that s_{T1}^* is a best response to s_{T2}^*. If Player 1 is of type $d > 1$, then defect is a dominant strategy. Suppose next Player 1 is of type $d \leq 1$.

If the pair of actions in period 2 is (C, C), then Player 2's type must be below 1, and hence starting from period 3 Player 2 cooperates in every period. Therefore, cooperating in every period $t \geq 3$ is a best response for Player 1 of type $d \leq 1$. If the pair of actions in period 2 is other than (C, C), then regardless of his type Player 2 defects in every period t, $t \geq 3$. Hence defecting in every period t, $t \geq 3$ is a best response to s_{T2}^* for all types of Player 1.

Next consider period 2. Suppose the first-period pair of actions is (C, C). Then Player 2's type must be below d_T^*, and Player 2 cooperates in period 2. Therefore, cooperate in period 2 is a best response of Player 1 of type $d \leq 1$.

Suppose next the first-period pair of actions is (D, D), then regardless of his type Player 2 defects in period 2, and defecting in period 2 is a best response for all types of Player 1.

Suppose the first-period pair of actions is (D, C). Then Player 2's type must be below d_T^*. Since $d_T^* < m_T^*$ (Step 1), Player 2 cooperates in period 2. Cooperating in period 2 is therefore a best response action for Player 1 of type $d \leq 1$.

Suppose the first-period pair of actions is (C, D). If Player 1 cooperates in period 2, his expected payoff is

$$v_1(C)|_{(C,D)} := \frac{F_B(1) - F_B(d_T^*)}{1 - F_B(d_T^*)} \cdot 1 + \frac{F_B(1) - F_B(d_T^*)}{1 - F_B(d_T^*)} \cdot \frac{\Delta}{\delta} + \left(1 - \frac{F_B(1) - F_B(d_T^*)}{1 - F_B(d_T^*)}\right) \cdot d_1 \cdot \frac{\Delta}{\delta}, \tag{51}$$

where $\frac{F_B(1)-F_B(d_T^*)}{1-F_B(d_T^*)}$ is the conditional probability that Player 2 cooperates in period 2, given that he defected in period 1. If Player 1 of type d_1 defects in period 2, his expected payoff is

$$v_1(D)|_{(C,D)} := d_1 + d_1 \cdot \frac{\Delta}{\delta}. \tag{52}$$

It can be verified that $v_1(C)|_{(C,D)} \geq v_1(D)|_{(C,D)}$ if and only if

$$d_1 \leq \underbrace{\frac{\delta \cdot \big(F_B(1) - F_B(d_T^*)\big) + \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)\big)}{\delta\big(1 - F_B(d_T^*)\big) + \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)}}_{m_T^*}. \tag{53}$$

By (13), the RHS of (53) is equal to m_T^*. Therefore, if the first-period pair of actions is (C, D), it is a best response for Player 1 of type $d_1 \leq m_T^*$ to cooperate in period 2.

Next we examine the first period actions. If Player 1 of type d_1 plays C in the first period and follows s_{T1}^* then after, his expected payoff (given that Player 2 plays s_{T2}^*) is

$$v_1(C) := F_B(d_T^*) + \delta \cdot F_B(1) \cdot 1 + \Delta \cdot \big[F_B(1) \cdot 1 + (1 - F_B(1)) \cdot d_1\big]. \tag{54}$$

If Player 1 of type d_1 plays D in the first period and follows s_{T1}^* then after, he obtains

$$v_1(D) := d_1 + \delta \cdot \big[F_B(d_T^*) \cdot 1 + \big(1 - F_B(d_T^*)\big) \cdot d_1\big] + \Delta \cdot \big[F_B(d_T^*) \cdot 1 + \big(1 - F_B(d_T^*)\big) \cdot d_1\big]. \tag{55}$$

By (54) and (55), it can be verified that $v_1(C) \geq v_2(D)$ if and only if

$$d_1 \leq \underbrace{\frac{F_B(d_T^*) + \delta \cdot \big(F_B(1) - F_B(d_T^*)\big) + \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)\big)}{1 + \delta\big(1 - F_B(d_T^*)\big) + \Delta \cdot \big(F_B(1) - F_B(d_T^*)\big)}}_{d_T^*}. \tag{56}$$

Note that the RHS of (56) is d_T^* given by (12).

We conclude that in the first period, it is a best response for Player 1 of type $d_1 \leq d_T^*$ to cooperate and all other types to defect. This completes the proof of Step 2.

Step 3: In this step we show that if F_1 satisfies (14), then for $F_B \in \mathscr{F}(F_1)$, d_T^* converges to 1 as $B \downarrow 1$.

Suppose that $F_1 \in \mathcal{G}_1$ and F_1 satisfies (14). Then there exists $\widehat{x} \in (0, 1)$ such that for all $x \in (\widehat{x}, 1)$,

$$F_1(x) > \overbrace{\frac{x - (\delta + \Delta)(1 - x)}{1 - (\delta + \Delta)(1 - x)}}^{R_T(x)}. \tag{57}$$

Let

$$H_1^T(x) := \frac{F_1(x) + (\delta + \Delta)\big(1 - F_1(x)\big)}{1 + (\delta + \Delta)\big(1 - F_1(x)\big)}.$$

It can be verified that the condition $F_1(x) > R_T(x)$ is equivalent to $H_1^T(x) > x$. Let

$$H_B^T(x) := \frac{F_B(x) + \delta\big(F_B(1) - F_B(x)\big) + \Delta\big(F_B(1) - F_B(x)\big)}{1 + \delta\big(1 - F_B(x)\big) + \Delta\big(F_B(1) - F_B(x)\big)}. \tag{58}$$

By (12), the threshold for cooperation in the first period of $G_T(F_B)$, d_T^*, is a fixed point of H_B^T.

It can be verified that for $B > 1$, $H_1^T(x) > H_B^T(x)$ for every $x \in (0, 1]$. By the uniform continuity of $H_B(x)$ as a bivariate function of (B, x) on $[1, 2] \times [0, 1]$, for every ζ, there exists $B^\zeta > 1$ such that for every $1 < B \leq B^\zeta$,

$$\max_{x \in [0,1]} \Big(|H_1^T(x) - H_B^T(x)|\Big) < \zeta. \tag{59}$$

To complete the proof of Proposition 6, let $0 < \epsilon < 1$. Since F_1 satisfies (14), there exists x_ϵ, $1 - \epsilon < x_\epsilon < 1$ such that $H_1^T(x_\epsilon) > x_\epsilon$. By (59), there exists B_ϵ such that for every $1 < B < B_\epsilon$, $H_B^T(x_\epsilon) > x_\epsilon$. By (58), $H_B^T(1) < F_B(1) < 1$. Since $H_B^T(x)$ is continuous in x, the equation $H_B^T(x) = x$ has a solution $d_T^* \in (1 - \epsilon, 1)$. Therefore, the first-period threshold d_T^* given in (12), is in $(1 - \epsilon, 1)$, as claimed.

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