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Patent licensing, entry and the incentive to innovate[☆]

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ABSTRACT

We analyze the economic impact of process innovations where the innovator auctions off licenses to both potential entrants and incumbent firms. It is shown that opening the market to entrant licensees, the incentive to innovate is maximized if the industry is initially a monopoly, as was envisioned by Schumpeter (1942). This is in contrast to previous literature on licensing of process innovations when entry is excluded: the incentive to innovate is maximized in an oligopoly market if licenses are sold by auction (Sen and Tauman, 2007) or in a competitive market if licenses are sold by royalty (Arrow, 1962). The post-innovation market structure, the diffusion of the innovation and the social welfare are analyzed and compared with the case where entry is excluded.

1. Introduction

The analysis of optimal licensing strategies of an innovator, the post innovation market structure as well as the incentive to innovate has been extensively studied in the literature and can be traced back to Arrow (1962). Arrow shows that the revenue of

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an innovator who sells licenses by means of a per-unit royalty is higher in a competitive market, compared with a monopolistic market. By focusing on the comparison of a monopolistic and a competitive market, Arrow avoids the strategic interactions between potential licensees, which is crucial in the analysis of an oligopoly. The game-theoretic framework which enables such interaction was introduced independently by Katz and Shapiro (1985) and Kamien and Tauman (1986) (see (Kamien, 1992) for a review of the first decade results on this topic). Kamien and Tauman (1986) and the extended analysis in Sen and Tauman (2007) show that an oligopoly (instead of a monopolist or a competitive market) maximizes the revenue of an innovator who sells licenses by either a lump sum fee or by a two part tariff (a combination of a lump sum fee and a per unit royalty).[1] In these papers, as well as most other papers on optimal licensing of new innovations, it is assumed that incumbent firms are the only potential licensees. An important point not sufficiently perceived in the literature, and which constitutes our main subject, is that when the innovation reduces the previously high entry cost and makes entry profitable, the innovator may benefit from selling licenses to entrants as well as to incumbent firms.

For example in the US steel industry, the traditional technology (integrated mills) produces on a large scale that is typically economical to build in 2 million-ton per year annual capacity and up. The introduction of a new technology for producing steel, the minimill, requires a much smaller capacity (a typical size is 200,000 to 400,000 tons per year) and can be easily started and stopped on a regular basis. Collard-Wexler and De Loecker (2014) show that the introduction of minimill brought in entry into the steel market and led to a drop in the market share of the incumbent technology. Moreover, minimill is more efficient in the sense that its total factor productivity is at least as high as that of the old technology.[2]

This paper analyzes the economic impact of process innovations where the innovator can sell licenses to both potential entrants and incumbent firms. Licenses in our model are sold by auction aiming to maximize the revenue of the innovator. The post-innovation market structure, the diffusion of the innovation and the incentive to innovate are analyzed.

In contrast to the literature dealing with licensing of process innovation to incumbent firms only, it is shown, surprisingly at first glance, that opening the Cournot market to entrant licensees, the incentive to innovate is maximized if the industry is initially a monopoly. This is true for drastic as well as non-drastic innovations. If the initial industry exogenously had one less incumbent, the innovator could induce the same post-licensing market structure by selling the same number of total licenses but shifting one

[1] The optimal size of the oligopoly depends on the magnitude of innovation, demand intensity and the marginal cost of production.

[2] Another example is in the television industry. The traditional cable TV was introduced nearly a half century before the Internet. When the Internet was first developed, sending video through it was difficult as the result of limited bandwidth capacity. By the mid of the first decade in the 21st century the broadband price went dramatically down and as a result many Internet TV companies entered the television industry (e.g., Netfix and Amazon). The entry of Internet TV companies induced one quarter of households to abandon their cable subscriptions (see (Gali, 2017)).

license from incumbents to entrants. This yields the innovator a weakly higher revenue since it avoids an incumbent's "replacement effect". This result is consistent with the observation of Schumpeter (1942) that monopolistic industries, those in which individual firms have a measure of control over their product price, provide a more hospitable atmosphere for innovation than purely competitive ones. This is also in line with Chen and Schwartz (2013), who show that the gain of a innovator from the exclusive use of a *product innovation* can be larger in a monopoly market than in a perfectly competitive one.

It is further shown that the post-innovation number of firms is larger the smaller is the magnitude of innovation. Namely less significant innovations diffuse more (i.e., are licensed to a larger number of firms). To clarify this point notice that the (negative) competition effect of an additional licensee on the innovator's revenue is increasing in the magnitude of the innovation and as a result the innovator is more reluctant to issue a large number of licenses for more significant innovations. Furthermore, it is shown that for relatively significant innovations the innovator chooses to sell licenses only to incumbent firms and not to entrants. For less significant innovations the innovator sells licenses to some entrants and to all incumbent firms. The result that the innovator prefers incumbent licensees to entrant licensees seems puzzling at first glance, since typically entrants are willing to pay for a license more than incumbent firms (each entrant is willing to pay all his profit while each incumbent firm is willing to pay only the incremental profit). However, an entrant licensee increases the number of active firms by 1 causing the Cournot profit of each firm to shrink. The effect of a weaker competition on the revenue of the innovator is larger and the innovator prefers incumbent firms to entrants.

We are aware of only few papers which consider entrants as potential licensees. Gilbert and David (1982) (GN hereafter) study a monopolistic market and show that the monopolist has an incentive to maintain its monopoly power by patenting new technologies to preempt potential competition. This leads to patents that are neither used nor licensed to others (shelving).[3] While GN analyze the interaction between a monopolistic incumbent and a potential entrant competing for an innovation, this paper studies the interaction between multiple incumbent firms and potential entrants where an outside innovator sells multiple licenses.

Hoppe et al. (2006) (HJM hereafter) is the paper closest to ours as it considers a market with several incumbent firms and many potential entrants. In HJM an innovator sells licenses through a uniform auction (UA), where he chooses a number k of licenses to sell to both incumbent firms and entrants. Each one of the k highest bidders (whether incumbent firm or entrant) obtains a license and all licensees pay the same amount, the $(k+1)$th highest bid. Externality plays an important role in UA, since each bidder's

[3] In a recent paper, Tauman et al. (2017) show that shelving may *not* occur under a monopolistic market if the innovator cannot give a take-it-or-leave-it offer, but rather engage in a bargaining process with the incumbent. The innovator may benefit from selling a few licenses to entrants before approaching the incumbent because it allows the innovator to collect fees from entrants during the bargaining process and not less important will credibly increase the innovator's threat on the incumbent, as it increases the number of licenses the innovator credibly sells in case the bargaining fails.

willingness to pay for a license depends not only on the total number of licenses, but also on the distribution of entrants and incumbent licensees. Unfortunately, UA has multiple equilibrium outcomes which results in multiple equilibrium payoffs of the innovator. In fact, for any number of licenses, $k \geq 1$, every partition (k_1, k_2) of k can be supported as an equilibrium outcome, where k_1 is the number of incumbent licensees and k_2 is the number of entrant licensees. The innovator only controls k and has no control over the partition of k into incumbent and entrant licensees. It is, therefore, difficult (if not impossible) to predict the economic impact of the innovation under UA. HJM show that the multiplicity of equilibrium outcomes in UA remains a problem even if (weakly) dominated strategies are eliminated. HJM therefore mostly deal with the sale of an exclusive license as well as some special cases involving multiple licenses and focus on whether entrants can be winners of licenses.

In this paper, we use, in addition to UA, an alternative auction mechanism, the non-uniform auction (NUA) which results in a unique equilibrium. In NUA the innovator chooses in addition to k, the exact partition (k_1, k_2). The winners of the auctions are the k_1 highest incumbent bidders and the k_2 highest entrant bidders. Each incumbent licensee pays the $(k_1 + 1)$th highest bid among the incumbents' bids while each entrant licensee pays the $(k_2 + 1)$th highest bid among the entrants' bids.[4] In contrast to UA, externalities do not play a role in NUA since the post-innovation market structure is fixed regardless of the bids. For every (k_1, k_2), the equilibrium outcome (in undominated strategies) of the auction is uniquely determined.

At first glance it seems that the ability to choose (k_1, k_2) and to differentiate the license fee of entrants from incumbent firms must yield the innovator a higher payoff in NUA (than in UA). But this may not be the case. Indeed, on one hand every entrant licensee pays in NUA her entire profit while in UA it is (like any incumbent licensee) only the incremental profit of an incumbent licensee. However, an incumbent is willing to pay more in UA as a licensee if he takes the place of an entrant licensee and thereby limits entry (preemption effect). In contrast, in NUA the preemption effect is absence since every incumbent licensee takes the place of another incumbent firm and thus does not change the number of active firms. It is shown that for relatively significant innovations, the innovator's highest equilibrium payoff in UA exceeds his unique equilibrium payoff in NUA. As mentioned before, for significant innovations, the innovator is best off having a small (or even 0) number of entrant licensees in both types of auctions. In this case the benefit from a higher willingness to pay of incumbent firms (preemption effect) exceeds the loss in revenue due to the inability to price differentiate entrant licensees.

This paper is organized as follows. In Section 2 we describe the model. In Section 3 we analyze the uniform auction (UA) and show that UA has multiple equilibrium outcomes. In Section 4 we study an alternative auction mechanism, the non-uniform auction (NUA)

[4] Another auction mechanism, semi-uniform auction (SUA), with a weaker asymmetry requirement than NUA is briefly discussed in Section 5. In SUA, the innovator, like in NUA, chooses both k and the partition (k_1, k_2) of k. The winners of the auctions are, again, the k_1 highest incumbent bidders and the k_2 highest entrant bidders. But unlike NUA, the license fee is the same across all licensees.

and use NUA to analyze the post-innovation market structure, the diffusion of the innovation and the incentive to innovate. We close with conclusion and extensions in Section 5.

2. The Model

Consider an industry with a set $N = \{1, \ldots, n\}$ of incumbent firms who produce the same product with marginal cost $c > 0$. Potential entrants are unable to enter the market either because of high entry cost or since the current technology is protected by a patent. An outside innovator comes along with an innovation which eliminates the entry cost and reduces the constant per unit cost from c to $c - \epsilon$, $0 < \epsilon \leq c$.[5] The number of potential entrants is assumed to be sufficiently large that exceeds the optimal number of licenses sold by the innovator.

Remark. The assumption that the new technology has zero entry cost is made for simplicity only. Our main insights remain valid as long as the entry cost with the new technology is relatively small.

The inverse demand function is assumed to be linear,[6] $p = \max(a - Q, 0)$, $a > c$. After the licensing stage, denote by $m_0(\leq n)$ the number of firms producing at the old marginal cost c (the non-licensee incumbent firms) and by $m_1(= k)$ the number of firms producing at the new marginal cost $c - \epsilon$ (the incumbent and entrant licensees). Denote by $\pi_0(m_0, m_1)$ the Cournot profit of a firm producing with the old technology and by $\pi_1(m_0, m_1)$ the Cournot profit of a firm producing with the new technology.

It can be verified that

$$\pi_0(m_0, m_1) = \begin{cases} \left(\frac{(a-c)-\epsilon m_1}{m_0+m_1+1}\right)^2 & \text{if } m_1 \leq \frac{a-c}{\epsilon} \\ 0 & \text{if } m_1 > \frac{a-c}{\epsilon} \end{cases} \tag{1}$$

$$\pi_1(m_0, m_1) = \begin{cases} \left(\frac{(a-c)+(m_0+1)\epsilon}{m_0+m_1+1}\right)^2 & \text{if } m_1 \leq \frac{a-c}{\epsilon} \\ \left(\frac{(a-c)+\epsilon}{m_1+1}\right)^2 & \text{if } m_1 > \frac{a-c}{\epsilon} \end{cases}$$

Without loss of generality we normalize $a - c$, the quantity demanded at the price c, to be 1. Note that the innovation is drastic iff $\epsilon \geq a - c$. Therefore, after normalization, the innovation is drastic if $\epsilon \geq 1$ and non-drastic if $\epsilon < 1$.

In UA the players are engaged in a three-stage game, G_u. In the first stage the innovator chooses and announces the number k of licenses to be auctioned off, to both incumbent firms and new entrants. In the second stage the licenses are allocated to the winners of a uniform auction where each one of the k highest bidders obtains a license and pays the $(k + 1)$th highest bid. In case of a tie, it is assumed that incumbent firms

[5] In principle even $\epsilon \leq 0$ may be valuable, if the innovation allows for a profitable entry. In this paper, however, we focus on the case $\epsilon > 0$.

[6] One of our main results which shows that the innovator's incentive to innovate is maximized when the market is initially a monopoly is true for a general demand structure.

have priority over entrants.[7] If a tie involves only one type of bidders, the tie is resolved at random. In the third and last stage the incumbents and the entrant licensees compete à la Cournot. Let $G_u(k)$ be the subgame of G_u which starts after the announcement of k.

In NUA the players are also engaged in a three-stage game, G_{nu}. In the first stage the innovator chooses and announces (k_1, k_2), where $0 \leq k_1 \leq n-1$ and $k_2 \geq 0$ are the number of licenses he auctions off to incumbent firms and entrants, respectively. Let $G_{nu}(k_1, k_2)$ be the subgame of G_{nu} which starts in the second stage of G_{nu}. In $G_{nu}(k_1, k_2)$, licenses are sold through a non-uniform auction. Each of the k_1 highest incumbent bidders obtains a license and pays the (k_1+1)th highest bid among the incumbents' bids. Similarly, each of the k_2 highest entrant bidders obtains a license and pays the (k_2+1)th highest bid among the entrants' bids. Ties are resolved at random. In the third stage the firms in the industry (licensees and non-licensees) engage in Cournot competition. Note that the auction is not well defined for $k_1 = n$. Thus we limit k_1 to $n-1$.

In $G_{nu}(k_1, k_2)$, the value of a license is uniquely determined for each bidder. This is not the case in UA where the value of a license typically depends on the distribution of incumbent and entrant licensees. Note that bidders do not usually have dominant strategies in UA.

Proposition 1. *Suppose bidders do not use dominated strategies. (i) If the innovator auctions off a total of $\frac{1}{\epsilon}$ licenses (using either UA or NUA), then the Cournot price is c, the pre-innovation marginal cost, and every non-licensee firm is driven out of the market. Each licensee pays his entire profit and the innovator obtains the total industry profit. (ii) It is never optimal for the innovator in both UA and NUA to auction off more than $\frac{1}{\epsilon}$ licenses.*

Proof. Part (i) is a straightforward consequence of (1). Part (i) asserts that when $k = \frac{1}{\epsilon}$ only licensees are active firms in the market, and this is obviously true for all $k \geq \frac{1}{\epsilon}$. Since the total industry profit is decreasing in k for $k \geq \frac{1}{\epsilon}$, part (ii) follows. $\square$

It will be shown (see Propositions 3 and A.6, below) that for $\epsilon > \frac{2}{n+1}$ the optimal number of licenses for the innovator is $k = \frac{1}{\epsilon}$ in both UA and NUA .

Remark. Here we treat the number of licenses as a continuous variable, which is a simplification assumption that is standard in the patent licensing literature (see (Kamien, 1992)). Our main insights remain valid if, instead, we assume that when the integer

[7] Let T_i (or T_e) be the set of incumbent (or entrant) bidders who bid the lowest winning bid $\underline{b}$. Let $|T_i|$ and $|T_e|$ be the number of firms in T_i and T_e, respectively. Let h be the number of bidders bid above $\underline{b}$. Each of the h highest bidders (whether an incumbent or an entrant) obtains a license. The allocation of the remaining $k-h$ licenses uses the following tie-breaking rule: if $k-h \leq |T_i|$, $k-h$ incumbents are chosen at random from T_i to be licensees. if $k-h > |T_i|$, each incumbent in T_i obtains a license and $k-h-|T_i|$ entrants are chosen at random from T_e to be licensees. The priority on incumbent firms can be justified by cheaper training and installation costs of the new technology (although negligible compared with the license fee) for incumbent licensees than for entrant licensees.

constraint is violated, the number of licenses awarded will be the next higher integer (the detailed analysis, however, is much more tedious).

When $\epsilon \geq 1$ (drastic innovation), even if the innovator sells an exclusive license, every non-licensee firm is driven out of the market and the only licensee will achieve the maximum industry profit – the monopolistic profit with the new technology. The innovator extracts the entire profit from this licensee via competitive bidding for the license. When $\epsilon < 1$ (non-drastic innovation), the pre-licensing market price is above c, and the post-licensing market price is below c if and only if the innovator auctions off more than $\frac{1}{\epsilon}$ licenses (regardless of the identity of the winners). Therefore when the innovator auctions off $k \geq \frac{1}{\epsilon}$ licenses, he obtains the entire industry profit. Since the total industry profit is decreasing in the number of active firms, selling more than $\frac{1}{\epsilon}$ licenses would reduce the industry profit compared with selling only $k = \frac{1}{\epsilon}$ licenses. The rest of the paper focuses on non-drastic innovations ($\epsilon < 1$) and on the case where in both UA and NUA the total number k of licenses does not exceed $\frac{1}{\epsilon}$.

3. Uniform auction

Consider the subgame $G_u(k)$ of G_u, for $1 \leq k \leq \frac{1}{\epsilon}$. Suppose (k_1, k_2) is an equilibrium outcome of $G_u(k)$, where k_1, $0 \leq k_1 \leq n$, is the number of incumbent licensees and $k_2 = k - k_1$ is the number of entrant licensees. Let $b_{(i)}$ be the ith highest bid in UA ($b_{(i)} = b_{(i+1)}$ if more than one bidders bid $b_{(i)}$).

The willingness to pay of an incumbent firm, i, for a license is the difference between his profit $\pi_1(n-k_1, k)$ as a licensee and his profit as a non-licensee if he drops out. The latter depends on the type of licensee replacing i. If it is an incumbent, the willingness to pay of i is

$$w_{il}^k(k_1) = \pi_1(n-k_1, k) - \pi_0(n-k_1, k). \tag{2}$$

If i expects to be replaced by an entrant, the total number of firms increases by 1 and the willingness to pay of i is

$$w_{ih}^k(k_1) = \pi_1(n-k_1, k) - \pi_0(n-k_1+1, k). \tag{3}$$

Note that w_{ih}^k can be regarded as an incumbent's willingness to pay for limiting entry and for using the superior technology. Since $\pi_0(n-k_1+1, k) \leq \pi_0(n-k_1, k)$ (see (1)), $w_{ih}^k(k_1) \geq w_{il}^k(k_1)$ (by Proposition 1 the equality holds only when $k = \frac{1}{\epsilon}$). The fact that the incumbent is willing to pay for a license more than his value from using the superior technology reflects the incumbent's incentive to *preempt entry*.

The willingness to pay of an entrant for a license is simply his Cournot profit,

$$w_e^k(k_1) = \pi_1(n-k_1, k). \tag{4}$$

For every k_1, $w_e^k(k_1) \geq w_{ih}^k(k_1) \geq w_{il}^k(k_1)$, and the equality holds only when $k = \frac{1}{\epsilon}$. That is, if $k < \frac{1}{\epsilon}$, for any fixed post-auction market structure (k_1, k_2), any entrant licensee is willing to pay for a license more than any incumbent licensee ($w_e^k(k_1) > w_{ih}^k(k_1)$). This reflects Arrow's "replacement effect". That is, without any strategic consideration, the incumbent gains less from the innovation than an entrant because the incumbent "replaces" his technology with a more efficient one, while the entrant had nothing to "rest on".

Nevertheless it is still possible that in equilibrium some incumbent firm wins a license. To clarify this point note that the post-auction market structure changes when an entrant outbids an incumbent licensee (the number of active firms increases by 1). Even though $w_e^k(k_1) > w_{ih}^k(k_1)$, it is possible that $w_{ih}^k(k_1) > w_e^k(k_1 - 1)$. That is, the incumbent's incentive to prevent entry may be larger than the entrant's gain when he becomes a licensee. For $n = 1$ (monopolist incumbent) and $k = 1$ (single license), since the industry profit is maximized under monopoly,

$$w_{ih}^1(1) = \pi_1(0, 1) - \pi_0(1, 1) > \pi_1(1, 1) = w_e^1(0).$$

If bidders do not use dominated strategies, the monopoly incumbent wins the auction and entry does not occur – this phenomenon is first discussed in Gilbert and David (1982). Hoppe et al. (2006) further analyze the market with more than one incumbent firms and show that entry may *not* be preempted in this case. This is because each incumbent may rely on the other to deter entry – as a result no one really carries out such action (bidders are required to use mixed strategies to support such equilibrium). The next proposition shows that, in fact, if bidders are *not* restricted to use undominated strategies, then for any given k, every distribution (k_1, k_2), $k_1 + k_2 = k$, can be supported as an equilibrium outcome. Our result does not require bidders to use mixed strategies. The multiplicity arises because each bidder's willingness to pay depends on his expectation about the post-auction market structure and the variation in such beliefs supports different equilibrium behaviors.

Proposition 2. *Let* $1 \le k < \frac{1}{\epsilon}$. *Then (i) any* (k_1, k_2), $0 \le k_1 \le n$ *and* $k_2 \ge 0$ *s.t.* $k_1 + k_2 = k$, *is an equilibrium outcome of* $G_u(k)$. *(ii) For* $k_1 = 0$, π *is an equilibrium payoff of the innovator in* $G_u(k)$ *if and only if* $\pi \in [0, kw_e^k(0)]$. *(iii) For* $1 \le k_1 \le n$, π *is an equilibrium payoff of the innovator in* $G_u(k)$ *if and only if* $\pi \in [0, kw_{ih}^k(k_1)]$.

Proof. See A.2 of the Appendix. □

Proposition 2 asserts that there are two types of multiplicity in $G_u(k)$ $(1 \le k < \frac{1}{\epsilon})$. First, every (k_1, k_2) s.t. $k_1 + k_2 = k$ is an equilibrium outcome of $G_u(k)$. Second, every (k_1, k_2) generates for the innovator a continuum of equilibrium payoffs. In fact, if in equilibrium the highest $k + 1$ bids are $\left(b^*_{(1)}, ..., b^*_{(k)}, b^*_{(k+1)}\right)$ then for every b, $0 \le b \le b^*_{(k+1)}$, changing only the $(k + 1)$th highest bid from $b^*_{(k+1)}$ to b also constitutes an equilibrium outcome.

The multiplicity of equilibrium outcomes remains a problem even if (weakly) dominated strategies are eliminated. Hoppe et al. (2006) deal with the game G_u with this restriction and show how complicated is the equilibrium analysis. The unfortunate conclusion is that there is no obvious way to predict the outcome of G_u nor the choice k of the innovator. To have some UA benchmark (which will be compared with the innovator's payoff in NUA), we focus here on a specific type of equilibrium in UA - the one that for any k yields the innovator the highest payoff in $G_u(k)$.

Proposition 3. *Suppose the innovator obtains for every k the highest equilibrium payoff in $G_u(k)$. (i) The corresponding equilibrium number of licensees in G_u is*

$$k_u^*(n,\epsilon) = \begin{cases} n+1 & \text{if } 0 < \epsilon < g(n) \\ n & \text{if } g(n) \le \epsilon \le f(n) \\ \tilde{k}(n,\epsilon) & \text{if } f(n) < \epsilon < \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \le \epsilon < 1. \end{cases}$$

(ii) If $0 < \epsilon < g(n)$ all licensees are entrants and if $g(n) \le \epsilon < 1$ all licensees are incumbent firms.

The formulas of $f(n)$, $g(n)$ and $\tilde{k}(n,\epsilon)$ are complicated and appear in A.1 of the Appendix. For $\epsilon \in [g(n), 1)$, $k_u^*(n,\epsilon)$ is continuous and decreasing in ϵ. It worth notice that $g(n) > 0$ only for $n \ge 4$.

Proof. See A.3 of the Appendix. □

Proposition 3 asserts that for less significant innovations the innovator obtains the highest equilibrium payoff when all licensees are entrants. On one hand an entrant licensee increases the number of active firms by 1 causing the Cournot profit of each firm to shrink. However when selling licenses only to entrants each licensee pays her entire Cournot profit for a license as opposed to the case where the innovator sells some licenses to incumbent firms (in the later case every licensee pays only the incremental profit of an incumbent licensee). When the magnitude of the innovation is relatively small the willingness to pay of an incumbent firm for a license is small and the negative effect of additional entry is offset by the incremental willingness to pay of entrants, as compared with incumbent firms. For relatively significant innovations, the benefit from having only entrant licensees does not compensate for the loss caused by a stronger competition and the innovator is best off when all licensees are incumbent firms.

Remark. Let $\frac{2}{n+1} \le \epsilon < 1$ and suppose bidders do not use dominated strategies. Then the unique optimal strategy of the innovator is to auction off $k = \frac{1}{\epsilon}$ licenses. The Cournot price reduces to the pre-innovation marginal cost c, and every non-licensee firm is driven out of the market. Consequently, the multiplicity of equilibrium points of UA occurs only for less significant innovations, $\epsilon < \frac{2}{n+1}$.

Denote by $\pi_u^*(n, \epsilon)$ the highest equilibrium payoff of the innovator in G_u (see formula in A.5 of the Appendix). We use $\pi_u^*(n, \epsilon)$ as a benchmark to compare the innovator's equilibrium payoff in UA with that obtained in NUA.

4. Non-uniform auction

In this section the innovator can choose and announce the number of licenses to be sold to incumbent firms $(0 \le k_1 \le n-1)$[8] and the number of licenses to be sold to potential entrants $(k_2 \ge 0)$. Each incumbent licensee pays the (k_1+1)th highest bid among the incumbents' bids. Each entrant licensee pays the (k_2+1)th highest bid among the entrants' bids. In $G_{nu}(k_1,\ k_2)$, since incumbents and entrants bid on separate auctions, each incumbent cannot prevent entry and therefore his willingness to pay is $w_{il}(k_1, k_2) = \pi_1(n - k_1, k_1 + k_2) - \pi_0(n - k_1, k_1 + k_2)$. The willingness to pay of each entrant is $w_e(k_1, k_2) = \pi_1(n - k_1, k_1 + k_2)$. Since bidding the true valuation is a (weakly) dominant strategy for each bidder, it is assumed that bidders bid truthfully in NUA. The innovator's equilibrium payoff in $G_{nu}(k_1,\ k_2)$ is then uniquely determined and it is given by

$$\pi_{nu}(k_1, k_2) = k_1 w_{il}(k_1, k_2) + k_2 w_e(k_1, k_2). \tag{5}$$

The next proposition deals with the innovator's incentive to innovate.

Proposition 4. *Regardless of the demand structure, if the innovator sells licenses by NUA, his revenue is maximized if the industry is initially a monopoly.*

Proof. Suppose there are n, $n \ge 2$ incumbent firms. Denote by (k_1^*, k_2^*) the optimal licensing strategy in G_{nu}. Let $K_{nu}^* = k_1^* + k_2^*$. The innovator's highest payoff is

$$\alpha \equiv k_1^*\big(\pi_1(n - k_1^*, K_{nu}^*) - \pi_0(n - k_1^*, K_{nu}^*)\big) + k_2^* \pi_1(n - k_1^*, K_{nu}^*). \tag{6}$$

Suppose the market has one less incumbent. That is, there are only $n-1$ incumbents left in the market.

Case 1. $k_1^* \ge 1$. Using the licensing strategy $(k_1^* - 1, k_2^* + 1)$, the innovator obtains

$$\beta \equiv (k_1^* - 1)\big(\pi_1(n - k_1^*, K_{nu}^*) - \pi_0(n - k_1^*, K_{nu}^*)\big) + (k_2^* + 1)\pi_1(n - k_1^*, K_{nu}^*). \tag{7}$$

Clearly for $K_{nu}^* = \frac{1}{\epsilon}$, $\pi_0(n - k_1^*, K_{nu}^*) = 0$ and $\alpha = \beta$. For $K_{nu}^* < \frac{1}{\epsilon}$, $\beta > \alpha$.

Case 2. Suppose $k_1^* = 0$. Using the licensing strategy $(0, k_2^*)$, the innovator obtains

$$\gamma \equiv k_2^* \pi_1(n-1, k_2^*) \ge k_2^* \pi_1(n, k_2^*) \equiv \alpha. \tag{8}$$

Again for $k_2^* = \frac{1}{\epsilon}$, $\gamma = \alpha$. For $k_2^* < \frac{1}{\epsilon}$, $\gamma > \alpha$

[8] The auction is not well defined for $k_1 = n$, thus we limit k_1 to $n-1$. We could extend our definition to $k_1 = n$ if we allow the innovator to charge a fixed fee in this case. For $n = 1$ this fee should be $\pi_1(0, k) - \pi_0(1, k)$. The analysis of NUA with this extension is tedious (see Sen and Tauman (2007)) and it will not change the main results of this paper.

Combining Cases 1 and 2, if $K^*_{nu} < \frac{1}{\epsilon}$ the innovator extracts strictly higher revenue with $n-1$ than with n incumbent firms. For $K^*_{nu} = \frac{1}{\epsilon}$, when the number of incumbent firms is $n-1$ the innovator obtains a payoff which is at least as high as the case where the number of incumbent firms is n. Since this is true for all $n \geq 2$, the proof is complete. □

Note that Proposition 4 does *not* depend on the linear structure of the demand function. For any fixed post-auction market structure, because of the "replacement effect", an entrant licensee is willing to pay at least as much as an incumbent licensee (more if $k < \frac{1}{\epsilon}$, and the same if $k = \frac{1}{\epsilon}$). Note that the "preemption effect" in GN is absent in NUA, since incumbents and entrants bid on separate licenses. Therefore for any number of licenses (k_1, k_2), the innovator contemplates, if the initial industry exogenously had one less incumbent, the innovator could induce the same post-licensing market structure by selling the same number of total licenses but shifting one license from incumbents to entrants. This yields the innovator a weakly higher revenue since it avoids an incumbent's "replacement effect".[9]

Proposition 4 is in contrast to Arrow (1962), which asserts that the gain from a process innovation under a competitive market is higher compared with a monopolistic market. The reason for the contradicting conclusion derives from Arrow's assumption that entry is not possible and licenses can be sold only to those who are already in the market.[10] Kamien and Tauman (1986) and the extended analysis in Sen and Tauman (2007) take into account the strategic interaction between potential licensees (which is crucial in the analysis of oligopoly) and show that the revenue of an innovator who sells licenses by either a lump sum fee or by a two part tariff (a combination of a lump sum fee and a per unit royalty) is maximized in an oligopoly market of a size which depends on the magnitude of innovation, demand intensity and the marginal cost of production. But these papers also limit potential licensees to incumbent firms only.

We next characterize the equilibrium of G_{nu}. Since $k_1 \leq n-1$ the case $n = 1$ is trivial to analyze. The following analysis focuses only on $n \geq 2$. Note that if the total number of licensees is $\frac{1}{\epsilon}$, every non-licensee firm is driven out of the market and each licensee pays his entire Cournot profit (see Proposition 1). In this case any partition (k_1, k_2), $k_1 + k_2 = \frac{1}{\epsilon}$ yields the innovator the same payoff. To avoid multiple equilibrium outcomes in G_{nu} we assume that the innovator in this case chooses $k_1 = \frac{1}{\epsilon}$ and $k_2 = 0$.

[9] As pointed out by one referee, the innovator's revenue would be maximized with no incumbents at all (instead of an initial monopoly), because an entrant would then bid the entire monopoly profit under the new technology.

[10] In Arrow (1962), if the innovation is drastic, the innovator in a monopolistic market can extract only the increment of the monopoly's profit, while in a competitive market (since each firm originally earns zero profit) the innovator can extracts the entire monopoly profit under the new technology. If the innovation is non-drastic, the benefit from a process innovation comes solely from reducing variable cost and, hence, is proportional to the size of output, which is lower under monopoly than under competitive market. Note that Arrow's result for non-drastic innovation does not extend to *product innovations*, where the scale of output is no longer a sufficient statistic, as noted by Chen and Schwartz (2013) (they term the opposing force the "coordination effect" – a two-product monopolist can profitably coordinate the post-innovation prices, unlike a rivalrous firm.)

Proposition 5. *(i) For significant innovations ($\epsilon \geq \frac{1}{2n-4}$) the innovator sells licenses only to incumbent firms and not to entrants. For less significant innovations ($\epsilon < \frac{1}{2n-4}$) the innovator sells licenses to some entrants and to all (but one) incumbent firms. (ii) The post-innovation number of firms is larger the less significant is the magnitude of innovation.*

Proof. (i) The unique equilibrium strategy of the innovator is characterized in A.6 of the Appendix. For $n \geq 3$, the claims follow from (20) to (21), the inequality $\frac{1}{2n-4} \leq \frac{2}{3n-5}$, and from $k_1^*(n, \epsilon)$ being decreasing in ϵ. The case $n = 2$ is an immediate consequence of (22)–(23). (ii) See A.7 of the Appendix. ☐

Proposition 5 (i) asserts that for large innovations the innovator sells all licenses to incumbents, while for smaller innovations he sells $n-1$ licenses to incumbents and the remaining $K_{nu}^* - (n-1)$ licenses to entrants (here K_{nu}^* is the total number of licenses).[11] The result that the innovator prefers incumbents to entrants seems puzzling at first glance, since the preemption effect is absent in NUA (the post-auction market structure is fixed regardless of the bids), and each entrant is willing to pay his entire profit for a license while the incumbent is willing to pay only the increment. However, on the other hand an entrant licensee increases the number of active firms by 1 causing the Cournot profit of each firm to shrink. The effect of a weaker competition on the revenue of the innovator is larger and the innovator prefers incumbent firms to entrants.

Part (ii) of Proposition 5 states that smaller innovations diffuse more. To clarify this point notice that the (negative) competition effect of additional licensee on the innovator's revenue is increasing in the magnitude of the innovation. As a result the innovator is more reluctant to issue a larger number of licenses for larger innovations. In fact, for $n \geq 3$ and for relatively small ϵ the innovator sells $2n$ licenses ($n-1$ licenses to incumbent firms and $n+1$ licenses to new entrants). As ϵ grows the number of licenses decreases and as the innovation becomes closer to a drastic innovation ($\epsilon \to 1$), the innovator sells one license only.

We next compare the innovator's equilibrium payoff in UA and in NUA. At first glance it seems that the ability to separate entrants (who has a higher willingness to pay) from incumbent firms and hence charge them differently should yield the innovator in NUA a higher payoff than UA. However, an incumbent licensee is willing to pay more in UA if he takes the place of an entrant licensee and hence limit entry (in contrast, this preemption effect is absence in NUA since the post-auction market structure is fixed regardless of the bids). It is, therefore, not clear which mechanism serves better the innovator. Denote by $\pi_{nu}^*(n, \epsilon)$ the innovator's (unique) equilibrium payoff in G_{nu} (see formula in A.8 of the Appendix). The next proposition compares $\pi_{nu}^*(n, \epsilon)$ with $\pi_u^*(n, \epsilon)$, the innovator's *highest* equilibrium payoff in UA (see formula in A.5 of the Appendix).

[11] Note that if $n = 2$, $\epsilon < \frac{1}{2n-4}$ always holds and licenses are sold to one incumbent firm and to some entrants.

Proposition 6. *For* $n \geq 4$, $\pi^*_{nu}(n,\epsilon) > \pi^*_u(n,\epsilon)$ *iff* $\epsilon < h(n)$. *For* $n = 2$ *or* $n = 3$, $\pi^*_{nu}(n,\epsilon) \leq \pi^*_u(n,\epsilon)$ *for all* $0 < \epsilon < 1$.

The formula of $h(n)$ is complicated and appears in A.1 of the Appendix.

Proof. See A.9 of the Appendix. □

Consider the market with at least 4 incumbent firms. Proposition 6 asserts that for significant innovations the innovator can obtain a higher payoff in UA than in NUA. For $\epsilon \geq \frac{1}{2n-4}(> h(n))$, in both types of auctions the innovator is best off having only incumbent licensees (see Propositions 3 and 5) and hence the innovator strictly prefers (the highest equilibrium payoff in) UA. When ϵ decreases from $\frac{1}{2n-4}$ to 0 the innovator in NUA sells an increasing number of licenses to entrants in addition to all (but one) incumbent firms. This implies that the benefit from differentiating the license fee of entrants from incumbent firms is more significant for smaller ϵ. In particular, for $\epsilon < h(n)$ the benefit from price differentiating entrant licensees exceeds the loss in revenue due to lower willingness to pay of incumbent firms (absence of preemption effect), and the innovator is better off using NUA.

If the pre-innovation market is duopoly or triopoly, the highest equilibrium payoff in UA yields the innovator a (weakly) higher payoff compared with NUA, regardless of the magnitude of the innovation. Note that unlike UA, in NUA the innovator is constrained by $k_1 \leq n-1$. Intuitively, in UA an incumbent firm always faces the threat of being replaced by an entrant; while in NUA each incumbent firm obtains a license for certain if $k_1 = n$ and therefore his willingness to pay in this case is 0.[12] The constraint $k_1 \leq n-1$ in NUA always leaves at least one incumbent firm producing with the old technology and this has a negative competition effect on every licensee. The magnitude of this (negative) effect is larger when the number of incumbent firms is smaller and it dominates all other (positive) effects when $n = 2$ or 3.

Finally, even if $\pi^*_u(n,\epsilon) > \pi^*_{nu}(n,\epsilon)$, there may exist other equilibrium points in UA which yields the innovator a lower payoff than in NUA. We illustrate this in the next example.

Example: Suppose $\epsilon = 0.2$ and $n = 5$.[13] In NUA, the innovator's unique equilibrium payoff is $4\big(\pi_1(1,4) - \pi_0(1,4)\big) = 0.213$ and it is obtained when he auctions off 4 licenses only to incumbent firms. In UA the innovator's highest equilibrium payoff is $4\big(\pi_1(1,4) - \pi_0(2,4)\big) = 0.214$, and it is obtained when he auctions off 4 licenses and all winners happen also to be incumbent firms. To support this equilibrium in UA the 5th highest bid has to be submitted by entrants only. In this case each of the 4 incumbent licensees pays more in UA than in NUA in attempt to limit entry. However, there are other equilibrium points

[12] Even with a minimum reservation price, the innovator is still reluctant to choose $k_1 = n$ in NUA. The optimal reservation price is $\pi_1(0, n+k_2) - \pi_0(1, n-1+k_2)$. That is, when an incumbent drops out, no one replaces him and he reduces the number of licensees. This increases the value of the "outside option" which in turn reduces each incumbent's willingness to pay.

[13] Recall that we assume $a - c = 1$.

in UA which yields the innovator a much lower payoff. Following Proposition 2, there is an equilibrium in which all 4 winners are entrants. In this equilibrium the innovator obtains only $4\pi_1(5,4) = 0.194$.[14]

5. Conclusion and extensions

This paper analyzes the economic impact of process innovations where the innovator auctions off licenses to both potential entrants and incumbent firms. If the innovator uses the uniform auction (UA), he has control over the total number k of licenses but not over the final distribution of licensees. Each bidder's willingness to pay for a license depends not only on the number of licenses but also on his expectation about the post-innovation market structure (the number k_1 of incumbent licensees and the number k_2 of entrant licensees, $k_1 + k_2 = k$). The multiplicity of such beliefs supports multiple equilibrium behaviors, which makes it almost impossible to predict the outcome of UA. This motivates us to use an alternative auction mechanism, the non-uniform auction (NUA), where the innovator conducts two separate auctions to entrants and incumbent firms, respectively. Since the post-innovation market structure is deterministic, the equilibrium outcome of the auction is unique and as such provides sharp predictions – this allows us to study the economic impact of innovations in the presence of potential entrants.

The post-innovation market structure, the diffusion of the innovation and the incentive to innovate are analyzed under NUA. Since most literature on patent licensing assumes that incumbent firms are the only potential licensees, we next briefly compare our results with the case where entry is excluded.

Opening the market to entrant licensees, the incentive to innovate is maximized if the industry is initially a monopoly.[15] This is in contrast to the case where entry is excluded, where the innovator's incentive to innovate is maximized in an industry of size $n = \max\left(3, 2\sqrt{2 + \frac{1}{\epsilon}} - 1\right)$ (this result is in line with (Kamien et al., 1992), see A.10 of the Appendix for a proof). Not surprisingly, opening the market to entrants has positive effect on social welfare and it yields the innovator a higher revenue, compared with the case where entry is excluded. However, it is shown (in A.10 of the Appendix) that these differences become smaller (namely, excluding entrants becomes less "harmful") when the magnitude of innovation increases. In particular, for sufficiently significant innovations, the innovator chooses to sell licenses only to incumbent firms and not to entrants - in this case the exclusion becomes completely "*harmless*". The "preemption effect" described in Gilbert and David (1982) is absence in NUA, since the post-auction market structure is fixed regardless of the bids of incumbent firms. Still, the innovator of a relatively significant innovation prefers to sell licenses only to incumbents and not to entrants. This is because an entrant licensee increases the number of active firms by 1 causing the total industry profit and the Cournot profit of each firm to shrink.

[14] Moreover this is the highest equilibrium payoff of the innovator among all equilibrium points where all 4 licensees are entrants.
[15] This result is proved under general demand structure and it does not confine to linear demand.

Another related auction mechanism is the semi-uniform auction (SUA), which has a weaker asymmetry requirement than NUA. In SUA the innovator, like in NUA, chooses both k and the exact partition $(k_1,\ k_2)$ of k where the winners of the auction are the k_1 highest incumbent bidders and the k_2 highest entrant bidders (ties are resolved at random). But unlike NUA, the license fee is the same across all licensees. To ensure the participation of incumbent firms in SUA the license fee is set to be the lowest winning bid.[16]

Like in NUA, when selling licenses by SUA, a monopolistic market provides the highest incentive to innovate. Comparing the innovator's payoff, NUA always (irrespective of the number of incumbent firms and the magnitude of the innovation) yields the innovator a (weakly) higher payoff. This is obvious since NUA, like SUA, specifies the partition of licenses into entrants and incumbent firms but, in addition, allows the innovator to price differentiate entrant licensees.[17] More surprisingly, when the innovation is relatively small, NUA results in a higher diffusion of technology than SUA. This is because when the innovation is small, in NUA the innovator sells licenses to all (but one) incumbent firms as well as some potential entrants; while in SUA he sells licenses only to new entrants and *not* to incumbent firms. This difference stems from the fact that in SUA, as long as the innovator sells licenses to incumbent firms, every winner pays the incremental profit of an incumbent licensee (this increment decreases to zero as the magnitude of the innovation decreases to zero). In contrast, the innovator in NUA can extract entrant licensees' entire profits even in the presence of incumbent licensees. To conclude, the ability to price differentiate new entrant licensees has positive effect not only on the innovator's revenue but also on social welfare, as compared with SUA. The detailed analysis of SUA appears in A.13 of the Appendix.

Appendix

A.1

$$
\begin{aligned}
f(n) &= \frac{n^3+n^2+2n+4+\sqrt{n^6+8n^5+30n^4+56n^3+50n^2+20n+4}}{3n^4+8n^3+10n^2+4n-4}\\
\tilde{k}(n,\epsilon) &= \{2n^3\epsilon+10n^2\epsilon+16n\epsilon+4n+8\epsilon+6\\
&\quad -\sqrt{4n^6\epsilon^2+34n^5\epsilon^2+119n^4\epsilon^2+4n^4\epsilon+220n^3\epsilon^2+26n^3\epsilon+227n^2\epsilon^2+62n^2\epsilon+124n\epsilon^2+4n^2+64n\epsilon+28\epsilon^2+12n+24\epsilon+9}\}/\\
&\quad \{3(2n+3)\epsilon\}\\
g(n) &= \max\left(0,\ \frac{3n^4+6n^3+7n^2+4n-4-2\sqrt{n^8+9n^7+31n^6+49n^5+29n^4-9n^3-16n^2-4n}}{5n^5+15n^4+19n^3+9n^2+4}\right)\\
h(n) &= \max\left(0,\ \frac{n^4+n^3+2n^2+4n-\sqrt{3n^7+14n^6+18n^5+7n^4+24n^3+40n^2-16}}{2(n^5+2n^4+n^3+n^2+4n+4)}\right)\\
r(n) &= \begin{cases} e_1(n) & \text{if} n \geq 17\\ f_1^{-1}(n) & \text{if} 2 \leq n \leq 16 \end{cases}
\end{aligned}
$$

[16] If (as in UA) every licensee pays the highest losing bid, it may happen that the bids of all incumbent licensees fall below the $(k+1)$th highest bid. This is the case if the $(k+1)$th highest bid is submitted by an entrant who bids her entire industry profit. Such bid exceeds the willingness to pay of incumbent firms and incumbents are best off not participating in this auction.

[17] SUA also yields the innovator a (weakly) lower payoff than his highest payoff in UA. On one hand all winners in SUA pays a uniform price, like in UA. On the other hand because of the absence of the "preemption effect" each incumbent firm is willing to pay less in SUA.

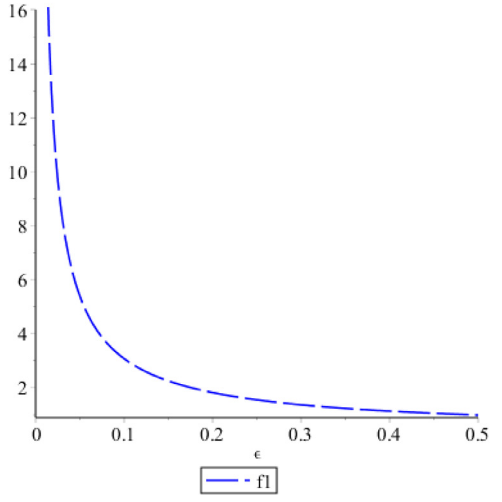

Fig. 1. The value of $f_1(\epsilon)$.

where

$$e_1(n) = \frac{3\,n - 5 - 2\,\sqrt{n^2 - 4\,n + 3}}{5\,n^2 - 14\,n + 13}$$

and

$$f_1(\epsilon) = -\frac{8\,\sqrt{1+2\,\epsilon}\epsilon^{3/2} - 11\,\epsilon^2 + 2\,\sqrt{-48\,\sqrt{1+2\,\epsilon}\epsilon^{7/2} - 12\,\sqrt{1+2\,\epsilon}\epsilon^{5/2} + 68\,\epsilon^4 + 34\,\epsilon^3 + 2\,\epsilon^2} - 3\,\epsilon}{\epsilon^2}$$

Fig. 1 shows that the inverse function of $f_1(\epsilon)$ exists for $0 < \epsilon < \frac{1}{2}$.

A.2. Proof of Proposition 2

(i) Let $k \geq 1$ and let $(k_1,\ k_2)$ s.t. $0 \leq k_1 \leq n-1$ and $k_2 = k - k_1$ (the case where $k_1 = n$ will be dealt separately). Let us show that $(k_1,\ k_2)$ is an equilibrium outcome of $G_u(k)$. Denote $b = \pi_1(n - k_1 - 1, k)$ (b is well defined since $k_1 \leq n-1$) and $\underline{b} = \pi_1(n - k_1, k) - \pi_0(n - k_1 + 1, k)$. Suppose that exactly k_1 incumbent firms and k_2 entrants bid b and only one entrant bids $\underline{\mathrm{b}}$. All other incumbents or entrants bid below $\underline{\mathrm{b}}$. Clearly $b_{(1)} = ... = b_{(k)} = b$, $b_{(k+1)} = \underline{b}$ and $\underline{\mathrm{b}} \leq b$. We claim that these bid profile constitutes an equilibrium of $G_u(k)$. Any incumbent licensee, i, obtains

$$\pi_1(n - k_1, k) - b_{(k+1)} = \pi_0(n - k_1 + 1, k).$$

If i lowers his bid below $\underline{\mathrm{b}}$ the entrant who bids $\underline{\mathrm{b}}$ will replace i. As a result there will be $n - k_1 + 1$ firms producing with the inferior technology and i will obtain $\pi_0(n - k_1 + 1, k)$, the same as his payoff as a licensee. Since the opportunity cost of any entrant is zero, an entrant licensee (when $k_2 \geq 1$) has no incentive to lower her bid. Next let us show that a non-licensee (incumbent or entrant) can not benefit from outbidding a licensee. Suppose j (incumbent or entrant) outbids a licensee i. Then he/she will increase the license fee from $\underline{b}$ to b. We claim that the industry profit of j is at most b and hence he has no incentive to become a licensee. Indeed, if both j and i are incumbent firms the industry profit of j as a licensee will be $\pi_1(n-k_1,k)$ which is smaller than $b=\pi_1(n-k_1-1,k)$. If j is an incumbent firm and i is an entrant, the number of firms using the inferior technology will reduce to $n-k_1-1$. The gross profit of j as a licensee will be b and his payoff, net of the new license fee, is zero. If j is an entrant, j will obtain an industry profit of $\pi_1(n-k_1+1,k)<b$ if i is an incumbent firm and $\pi_1(n-k_1,k)<b$ if i is an entrant. In both cases j's net payoff is negative. To complete the proof of part (i) suppose that $k_1=n$ and hence $k_2=k-n$. Suppose every incumbent firm and exactly k_2 entrants bid $b=\pi_1(0,k)$, one entrant only bids $\underline{b}=\pi_1(0,k)-\pi_0(1,k)$ and every other bidder bids below $\underline{b}$. The license fee is $\underline{b}$ and it is easy to verify that these bids constitute an equilibrium of $G_u(k)$.

(ii) Let $k_1=0$, $\tilde{b}\in[0,\pi_1(n,k)]$ and $b=\pi_1(n-1,k)$. Suppose exactly k entrants bid b, one entrant only bids $\tilde{b}$ and every other bidder bids below $\tilde{b}$. The license fee is $b_{(k+1)}=\tilde{b}$. Since $\pi_1(n,k)-\tilde{b}\geq 0$, no (entrant) licensee benefits from lowering his bid below $\tilde{b}$. Suppose next that a non-licensee j (incumbent or entrant), bids above b. Then the new license fee will increase to $b=\pi_1(n-1,k)$ and j's industry profit is $\pi_1(n-1,k)$ if j is an incumbent firm, and $\pi_1(n,\,k)$ if j is an entrant. In both cases the industry profit does not exceed the license fee. Finally, there is no equilibrium of $G_u(k)$ with $k_1=0$ and s.t. $b_{(k+1)}>\pi_1(n,k)$. Otherwise, the industry profit of a licensee does not cover the license fee.

(iii) Suppose $1\leq k_1\leq n-1$ and let $\tilde{b}\in[0,w_{ih}^k(k_1)]$, where by (3) $w_{ih}^k=\pi_1(n-k_1,k)-\pi_0(n-k_1+1,k)$. Denote $b=\pi_1(n-k_1-1,k)$. Suppose exactly k_1 incumbent firms and k_2 entrants bid b, one entrant only bids $\tilde{b}$ and every other bidder bids below $\tilde{b}$. Then $b_{(k+1)}=\tilde{b}$ is the license fee. An incumbent licensee obtains

$$\pi_1(n-k_1,k)-\tilde{b}\geq\pi_0(n-k_1+1,k). \tag{9}$$

If he lowers his bid below $\tilde{b}$ he will obtain $\pi_0(n-k_1+1,k)$. By (9) this will not benefit him. A non-licensee j (incumbent or entrant) who outbids a licensee i (incumbent or entrant) will increase the license fee from $\tilde{b}$ to $b=\pi_1(n-k_1-1,k)$. It is easy to verify that independently of the identity of j and i, j's industry profit will not exceed $\pi_1(n-k_1-1,k)$.

Next suppose $k_1=n$. Let $\tilde{b}\in[0,\pi_1(0,k)-\pi_0(1,k)]$ and let $b=\pi_1(0,k)$. Suppose every incumbent firm and exactly $k_2=k-n$ entrants bid b. Suppose also that only one entrant bids $\tilde{b}$ and all other bidders bid below $\tilde{b}$. Then the license fee is $b_{(k+1)}=\tilde{b}$. A licensee obtains

$$\pi_1(0.k)-\tilde{b}\geq\pi_0(1,k)\geq 0. \tag{10}$$

If an incumbent licensee lowers his bid below $\tilde{b}$ he will obtain $\pi_0(1,\,k)$ and by (10) he will not improve his payoff. If a non-licensee entrant j outbids a licensee i the new license fee will be $b = \pi_1(0, k)$ and again, independently of the identity of i, the industry profit of j will not exceed $\pi_1(0,\ k)$.

Finally, for $k \geq 1$ the willingness of an incumbent firm to pay for a license is at most $w_{ih}^k(k_1)$. Thus there is no equilibrium b^* of $G_u(k)$ s.t. $1 \leq k_1 \leq n$ and $b^*_{(k+1)} > w_{ih}^k(k_1)$.

A.3. Proof of Proposition 3

Lemma 1. *For any $1 \leq k \leq \frac{1}{\epsilon}$, the innovator's highest equilibrium payoff in $G_u(k)$ is obtained when either $k_1 = 0$ or $k_1 = \min(k, n)$.*

Proof. By Proposition 2, given an arbitrary $1 \leq k \leq \frac{1}{\epsilon}$, any $0 \leq k_1 \leq \min\,(k,\ n)$ can emerge as an equilibrium outcome. In addition the highest payoff of the innovator is $k\pi_1(n,\ k)$ if $k_1 = 0$ and $kw_{ih}^k(k_1)$ if $1 \leq k_1 \leq n$. It is shown in the Appendix (see A.4) that $w_{ih}^k(k_1)$ is increasing in k_1. Thus $kw_{ih}^k(k_1)$ is maximized when $k_1 = \min(k, n)$. □

By Lemma 1 the highest equilibrium payoff of the innovator in G_u is

$$\pi_u^*(n, \epsilon) = \max\left(\pi^0(n, \epsilon), \hat{\pi}(n, \epsilon)\right)$$

where $\pi^0(n, \epsilon) = \max_{k \geq 1} k\pi_1(n, k)$ and $\hat{\pi}(n, \epsilon) = \max_{k \geq 1} k(\pi_1(n - \min(k, n), k) - \pi_0(n - \min(k, n) + 1, k))$.

Let $k_u^*(n, \epsilon)$, $k^0(n,\ \epsilon)$ and $\hat{k}(n, \epsilon)$ be maximizers of $\pi_u^*(n, \epsilon)$, $\pi^0(n,\ \epsilon)$ and $\hat{\pi}(n, \epsilon)$, respectively. Clearly either $k^0(n,\ \epsilon)$ or $\hat{k}(n, \epsilon)$ is a maximizer of $\pi_u^*(n, \epsilon)$.

Lemma 2. *$k(\pi_1(0, k) - \pi_0(1, k))$ is decreasing in k.*

Proof. Let

$$J = k(\pi_1(0, k) - \pi_0(1, k)) = k\left(\left(\frac{1+\epsilon}{1+k}\right)^2 - \left(\frac{1-k\epsilon}{2+k}\right)^2\right)$$

Then

$$\frac{\partial J}{\partial k} = A\epsilon^2 + B\epsilon + C$$

where A, B and C are functions of k. In particular,

$$A = -\frac{k^6 + 9\,k^5 + 22\,k^4 + 24\,k^3 + 12\,k^2 - 4\,k - 8}{(1+k)^3(2+k)^3}$$

$$B = -\frac{-6\,k^4 - 14\,k^3 - 12\,k^2 - 16\,k - 16}{(1+k)^3(2+k)^3}$$

$$C = -\frac{4\,k^3 + 9\,k^2 + k - 6}{(1+k)^3(2+k)^3}$$

and

$$B^2 - 4AC = \frac{-16(k^3 + k^2 - 2k - 1)}{(k+2)^4(1+k)^2}$$

Clearly $A < 0$ for $k \geq 1$ and $B^2 - 4AC < 0$ for $k \geq 2$. Therefore $\frac{\partial J}{\partial k} < 0$ for $k \geq 2$ and J is maximized either at $k = 1$ or at $k = 2$.

Since $J|_{k=1} = \frac{1}{36}(\epsilon + 5)(5\epsilon + 1)$ and $J|_{k=2} = -\frac{1}{72}(10\epsilon + 1)(2\epsilon - 7)$, for every ϵ

$$J|_{k=1} - J|_{k=2} = \frac{5}{12}\epsilon^2 - \frac{2}{9}\epsilon + \frac{1}{24} > 0$$

Thus $J = k\big(\pi_1(0,k) - \pi_0(1,k)\big)$ is decreasing in k for $k \geq 1$. □

Lemma 3. *(i)* $k^0(n,\epsilon) \leq n+1$ *and (ii)* $\hat{k}(n,\epsilon) \leq n$.

Proof. (i) By (1) and Proposition 1

$$\pi^0(n,\epsilon) = \max_{1 \leq k \leq \frac{1}{\epsilon}} \frac{k\big(1 + (n+1)\epsilon\big)^2}{(n+k+1)^2}$$

and it is maximized at $k = \min(n+1, \frac{1}{\epsilon})$. Hence $k^0(n,\epsilon) \leq n+1$.

(ii) Let

$$\hat{\pi}_1(n,\epsilon) = \max_{1 \leq k < n} k\big(\pi_1(n-k,k) - \pi_0(n-k+1,k)\big)$$

and

$$\hat{\pi}_2(n,\epsilon) = \max_{k \geq n} k\big(\pi_1(0,k) - \pi_0(1,k)\big)$$

Then $\hat{\pi}(n,\epsilon) = \max\big(\hat{\pi}_1(n,\epsilon), \hat{\pi}_2(n,\epsilon)\big)$. But $k\big(\pi_1(0,k) - \pi_0(1,k)\big)$ is decreasing in k (Lemma 2). This implies $\hat{k}(n,\epsilon) \leq n$. □

The next lemma characterizes both $k^0(n, \epsilon)$ and $\hat{k}(n,\epsilon)$.

Lemma 4.

$$k^0(n,\epsilon) = \begin{cases} n+1 & \text{if } 0 < \epsilon < \frac{1}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{1}{n+1} \leq \epsilon < 1 \end{cases} \tag{11}$$

and

$$\hat{k}(n,\epsilon) = \begin{cases} n & \text{if } 0 < \epsilon < f(n) \\ \tilde{k}(n,\epsilon) & \text{if } f(n) \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} < \epsilon < 1 \end{cases} \tag{12}$$

where

$$f(n) = \frac{n^3 + n^2 + 2n + 4 + \sqrt{n^6 + 8n^5 + 30n^4 + 56n^3 + 50n^2 + 20n + 4}}{3n^4 + 8n^3 + 10n^2 + 4n - 4},$$

$$\tilde{k}(n,\epsilon) = \{2n^3\epsilon + 10n^2\epsilon + 16n\epsilon + 4n + 8\epsilon + 6 - \sqrt{4n^6\epsilon^2 + 34n^5\epsilon^2 + 119n^4\epsilon^2 + 4n^4\epsilon + 220n^3\epsilon^2 + 26n^3\epsilon + 227n^2\epsilon^2 + 62n^2\epsilon + 124n\epsilon^2 + 4n^2 + 64n\epsilon + 28\epsilon^2 + 12n + 24\epsilon + 9}\} / \{3(2n+3)\epsilon\}.$$

Here $\frac{1}{\epsilon} \leq \tilde{k}(n,\epsilon) \leq n$ *for* $f(n) \leq \epsilon \leq \frac{2}{n+1}$ *and* $\tilde{k}(n,\epsilon)$ *is decreasing in* ϵ.

Proof. (11) follows from the proof of part (i) of Lemma 3. We next analyze $\hat{k}(n,\epsilon)$.

By part (ii) of Lemma 3, $\hat{k}(n,\epsilon) \leq n$. Then

$$\begin{aligned}\hat{\pi}(n,\epsilon) &= \max_{1\leq k\leq n} k\big(\pi_1(n-k,k) - \pi_0(n-k+1,k)\big)\\ &= k\left(\frac{(1+(n-k+1)\epsilon)^2}{(n+1)^2} - \frac{(1-k\epsilon)^2}{(n+2)^2}\right)\end{aligned}$$

The first order condition is

$$\frac{\partial\hat{\pi}(n,\epsilon)}{\partial k} = Dk^2 + Ek + F \tag{13}$$

where

$$\begin{aligned}D &= \frac{6\,n\epsilon^2 + 9\,\epsilon^2}{(n+1)^2(n+2)^2} > 0,\\ E &= \frac{-4\,n^3\epsilon^2 - 20\,n^2\epsilon^2 - 32\,n\epsilon^2 - 8\,n\epsilon - 16\,\epsilon^2 - 12\,\epsilon}{(n+1)^2(n+2)^2},\\ F &= \frac{n^4\epsilon^2 + 6\,n^3\epsilon^2 + 2\,n^3\epsilon + 13\,n^2\epsilon^2 + 10\,n^2\epsilon + 12\,n\epsilon^2 + 16\,n\epsilon + 4\,\epsilon^2 + 2\,n + 8\,\epsilon + 3}{(n+1)^2(n+2)^2}\end{aligned}$$

and

$$\begin{aligned}E^2 - 4DF = 4\{&\epsilon^2(4\,n^6\epsilon^2 + 34\,n^5\epsilon^2 + 119\,n^4\epsilon^2 + 4\,n^4\epsilon + 220\,n^3\epsilon^2 + 26\,n^3\epsilon\\ &+ 227\,n^2\epsilon^2 + 62\,n^2\epsilon + 124\,n\epsilon^2 + 4\,n^2 + 64\,n\epsilon + 28\,\epsilon^2 + 12\,n + 24\,\epsilon + 9)\}/\\ &\{(n+1)^4(n+2)^4\} > 0\end{aligned}$$

Let c_1 and c_2 be the solution in k of the quadratic function $\frac{\partial\hat{\pi}(n,\epsilon)}{\partial k} = 0$. Then

$$\begin{aligned}c_1 = \{&2n^3\epsilon + 10n^2\epsilon + 16n\epsilon + 4n + 8\epsilon + 6\\ &-\sqrt{4n^6\epsilon^2 + 34n^5\epsilon^2 + 119n^4\epsilon^2 + 4n^4\epsilon + 220n^3\epsilon^2 + 26n^3\epsilon + 227n^2\epsilon^2 + 62n^2\epsilon + 124n\epsilon^2 + 4n^2 + 64n\epsilon + 28\epsilon^2 + 12n + 24\epsilon + 9}\}/\\ &\{3(2n+3)\epsilon\}\\ c_2 = \{&2n^3\epsilon + 10n^2\epsilon + 16n\epsilon + 4n + 8\epsilon + 6\\ &+\sqrt{4n^6\epsilon^2 + 34n^5\epsilon^2 + 119n^4\epsilon^2 + 4n^4\epsilon + 220n^3\epsilon^2 + 26n^3\epsilon + 227n^2\epsilon^2 + 62n^2\epsilon + 124n\epsilon^2 + 4n^2 + 64n\epsilon + 28\epsilon^2 + 12n + 24\epsilon + 9}\}/\\ &\{3(2n+3)\epsilon\}\end{aligned}$$

It can be easily verified that when $\epsilon \geq 0$ and $n \geq 1$, $c_1 > 0$. Next we compare c_1 with $\frac{1}{\epsilon}$.

$$\frac{1}{\epsilon} - c_1 = \frac{s(n,\epsilon) - t(n,\epsilon)}{3(2n+3)\epsilon} \tag{14}$$

where

$$s(n,\epsilon) = \sqrt{4n^6\epsilon^2 + 34n^5\epsilon^2 + 119n^4\epsilon^2 + 4n^4\epsilon + 220n^3\epsilon^2 + 26n^3\epsilon + 227n^2\epsilon^2 + 62n^2\epsilon + 124n\epsilon^2 + 4n^2 + 64n\epsilon + 28\epsilon^2 + 12n + 24\epsilon + 9}$$

and

$$t(n,\epsilon) = 2n^3\epsilon + 10n^2\epsilon + 16n\epsilon + 8\epsilon - 2n - 3 \tag{15}$$

For $\epsilon \geq 0$, $s(n,\epsilon) > 0$ and it can be easily verified that $t(n,\epsilon) \leq 0$ iff $\epsilon \leq \frac{2n+3}{2(n+1)(n+2)^2}$. By (14) for $\epsilon \leq \frac{2n+3}{2(n+1)(n+2)^2}$, $c_1 \leq \frac{1}{\epsilon}$. If, however, $\epsilon > \frac{2n+3}{2(n+1)(n+2)^2}$ by (15) $t(n,\epsilon) > 0$. It can be easily verified that

$$\big(s(n,\epsilon)\big)^2 \geq \big(t(n,\epsilon)\big)^2 \quad \text{iff} \quad 0 \leq \epsilon \leq \frac{2}{n+1} \tag{16}$$

and for all $n \geq 1$, in which case, again, $c_1 \leq \frac{1}{\epsilon}$. It can also be verified that $\frac{2}{n+1} > \frac{2n+3}{2(n+1)(n+2)^2}$ for $n \geq 1$. Therefore $c_1 > \frac{1}{\epsilon}$ iff $\epsilon > \frac{2}{n+1}$. Since the optimal k is bounded above by $\frac{1}{\epsilon}$ (Proposition 1), for $\epsilon > \frac{2}{n+1}$, $\hat{k}(n,\epsilon) = \frac{1}{\epsilon}$. Next we analyze the case $0 \leq \epsilon \leq \frac{2}{n+1}$ (or equivalently $c_1 \leq \frac{1}{\epsilon}$). We first compare the value of $\frac{1}{\epsilon}$ and c_2.

$$\frac{1}{\epsilon} - c_2 = \frac{-s(n,\epsilon) - t(n,\epsilon)}{3(2n+3)\epsilon}$$

as shown above, $t(n,\epsilon) \geq 0$ iff $\epsilon \geq \frac{2n+3}{2(n+1)(n+2)^2}$. For $\frac{2n+3}{2(n+1)(n+2)^2} \leq \epsilon \leq \frac{2}{n+1}$, $c_2 \geq \frac{1}{\epsilon}$. Since $(s(n,\epsilon))^2 \geq (t(n,\epsilon))^2$ for $0 \leq \epsilon < \frac{2n+3}{2(n+1)(n+2)^2}$, again $c_2 \geq \frac{1}{\epsilon}$. Thus for any $\epsilon \leq \frac{2}{n+1}$, $c_2 \geq \frac{1}{\epsilon}$. This together with (13) imply that $\hat{\pi}(n,\epsilon)$ is maximized at $k = c_1$.

Finally we compare the value of c_1 with n.

$$n - c_1 = \frac{s(n,\epsilon) - (2n^3\epsilon + 4n^2\epsilon + 7\epsilon n + 4n + 8\epsilon + 6)}{3(2n+3)\epsilon}$$

It can be easily verified that

$$\begin{aligned}
&\big(s(n,\epsilon)\big)^2 - (2n^3\epsilon + 4n^2\epsilon + 7\epsilon n + 4n + 8\epsilon + 6)^2 \\
&= (18n^5 + 75n^4 + 132n^3 + 114n^2 + 12n - 36)\epsilon^2 \\
&\quad - (12n^4 + 30n^3 + 42n^2 + 84n + 72)\epsilon - (12n^2 + 36n + 27)
\end{aligned}$$

Thus $c_1 \geq n$ iff the last term ≤ 0. The solution of this quadratic inequality is

$$\begin{aligned}
&\frac{n^3 + n^2 + 2n + 4 - \sqrt{n^6 + 8n^5 + 30n^4 + 56n^3 + 50n^2 + 20n + 4}}{3n^4 + 8n^3 + 10n^2 + 4n - 4} \\
&\leq \epsilon \leq \frac{n^3 + n^2 + 2n + 4 + \sqrt{n^6 + 8n^5 + 30n^4 + 56n^3 + 50n^2 + 20n + 4}}{3n^4 + 8n^3 + 10n^2 + 4n - 4} \equiv f(n)
\end{aligned}$$

It can be easily verified that $\frac{n^3+n^2+2n+4-\sqrt{n^6+8n^5+30n^4+56n^3+50n^2+20n+4}}{3n^4+8n^3+10n^2+4n-4} < 0$ for $n \geq 1$. It can also be verified that $f(n) \leq \frac{2}{n+1}$ for $n \geq 1$.

Consequently, for $0 \leq \epsilon \leq f(n)$, $n \leq c_1 \leq \frac{1}{\epsilon}$ and $\hat{k}(n,\epsilon) = n$; for $f(n) < \epsilon \leq \frac{2}{n+1}$, $c_1 < n$, $c_1 < \frac{1}{\epsilon}$ and $\hat{k}(n,\epsilon) = c_1$; for $\frac{2}{n+1} \leq \epsilon \leq 1$, $\frac{1}{\epsilon} < c_1 < n$ and $\hat{k}(n,\epsilon) = \frac{1}{\epsilon}$. It is left to show that c_1 is decreasing in ϵ. We first compute the first order derivative of c_1 with respect to ϵ.

$$\begin{aligned}\frac{\partial c_1}{\partial \epsilon} = &\{n^3\epsilon + 5n^2\epsilon + 8\epsilon n \\ &- 2\sqrt{4n^6\epsilon^2 + 34n^5\epsilon^2 + 119n^4\epsilon^2 + 4n^4\epsilon + 220n^3\epsilon^2 + 26n^3\epsilon + 227n^2\epsilon^2 + 62n^2\epsilon + 124n\epsilon^2 + 4n^2 + 64\epsilon n + 28\epsilon^2 + 12n + 24\epsilon + 9} \\ &+ 2n + 4\epsilon + 3\}/ \\ &\{3\epsilon^2\sqrt{4n^6\epsilon^2 + 34n^5\epsilon^2 + 119n^4\epsilon^2 + 4n^4\epsilon + 220n^3\epsilon^2 + 26n^3\epsilon + 227n^2\epsilon^2 + 62n^2\epsilon + 124n\epsilon^2 + 4n^2 + 64\epsilon n + 28\epsilon^2 + 12n + 24\epsilon + 9}\}\end{aligned}$$

It can be easily verified that $\frac{\partial c_1}{\partial \epsilon} < 0$ for $n \geq 1$ and $\epsilon > 0$. The proof of Lemma 4 is complete. $\square$

We are now ready to characterize the equilibrium number of licensees in G_u, for the "lucky" innovator.

Case 1: Suppose $0 \leq \epsilon \leq \min(\frac{1}{n+1}, f(n))$, then $k^0(n,\epsilon) = n+1$ and $\hat{k}(n,\epsilon) = n$.

$$\begin{aligned}\pi^0(n,\epsilon) - \hat{\pi}(n,\epsilon) &= (n+1)\pi_1(n, n+1) - n\big(\pi_1(0,k) - \pi_0(1,k)\big) \\ &= \frac{(5n^5 + 15n^4 + 19n^3 + 9n^2 + 4)\epsilon^2 - (6n^4 + 12n^3 + 14n^2 + 8n - 8)\epsilon + n^3 - 3n^2 - 4n + 4}{4(n+1)^2(n+2)^2}.\end{aligned} \tag{17}$$

It is easy to verify that $\pi^0(n,\epsilon) \leq \hat{\pi}(n,\epsilon)$ iff

$$\begin{aligned}&\frac{3n^4 + 6n^3 + 7n^2 + 4n - 4 - 2\sqrt{n^8 + 9n^7 + 31n^6 + 49n^5 + 29n^4 - 9n^3 - 16n^2 - 4n}}{5n^5 + 15n^4 + 19n^3 + 9n^2 + 4} \\ &\leq \epsilon \leq \frac{3n^4 + 6n^3 + 7n^2 + 4n - 4 + 2\sqrt{n^8 + 9n^7 + 31n^6 + 49n^5 + 29n^4 - 9n^3 - 16n^2 - 4n}}{5n^5 + 15n^4 + 19n^3 + 9n^2 + 4}\end{aligned}$$

Let $d_1 = \frac{3n^4+6n^3+7n^2+4n-4-2\sqrt{n^8+9n^7+31n^6+49n^5+29n^4-9n^3-16n^2-4n}}{5n^5+15n^4+19n^3+9n^2+4}$ and $d_2 = \frac{3n^4+6n^3+7n^2+4n-4+2\sqrt{n^8+9n^7+31n^6+49n^5+29n^4-9n^3-16n^2-4n}}{5n^5+15n^4+19n^3+9n^2+4}$. We next show that $d_1 < \min(\frac{1}{n+1}, f(n)) < d_2$. First observe that

$$d_2 - \frac{1}{n+1} = \frac{(n+1)\sqrt{n^8 + 9n^7 + 31n^6 + 49n^5 + 29n^4 - 9n^3 - 16n^2 - 4n} - (n^5 + 3n^4 + 3n^3 + 4 - n^2)}{\frac{1}{2}(5n^5 + 15n^4 + 19n^3 + 9n^2 + 4)(n+1)}.$$

It can be easily verified that $d_2 > \frac{1}{n+1}$ for $n \geq 1$. Thus $d_2 \geq \min(\frac{1}{n+1}, f(n))$. Next observe that

$$\frac{1}{n+1} - d_1 = \frac{(n+1)\sqrt{n^8 + 9n^7 + 31n^6 + 49n^5 + 29n^4 - 9n^3 - 16n^2 - 4n} + (n^5 + 3n^4 + 3n^3 + 4 - n^2)}{\frac{1}{2}(5n^5 + 15n^4 + 19n^3 + 9n^2 + 4)(n+1)} > 0$$

thus $d_1 < \frac{1}{n+1}$. The analytical comparison between the value of d_1 and $f(n)$ is complicated. The numerical comparison is shown in Fig. 2. Form the figure, d_1 (blue) is less than $f(n)$ for $1 \leq n \leq 100$.

Since $d_1 < \min(\frac{1}{n+1}, f(n)) < d_2$, for $0 \leq \epsilon < d_1$, $\pi^0(n,\epsilon) \geq \hat{\pi}(n,\epsilon)$ and $k_2^*(n,\epsilon) = k^0(n,\epsilon) = n+1$. For $d_1 \leq \epsilon \leq \min(\frac{1}{n+1}, f(n))$, $\pi^0(n,\epsilon) < \hat{\pi}(n,\epsilon)$ and $k_2^*(n,\epsilon) = \hat{k}(n,\epsilon) = n$.

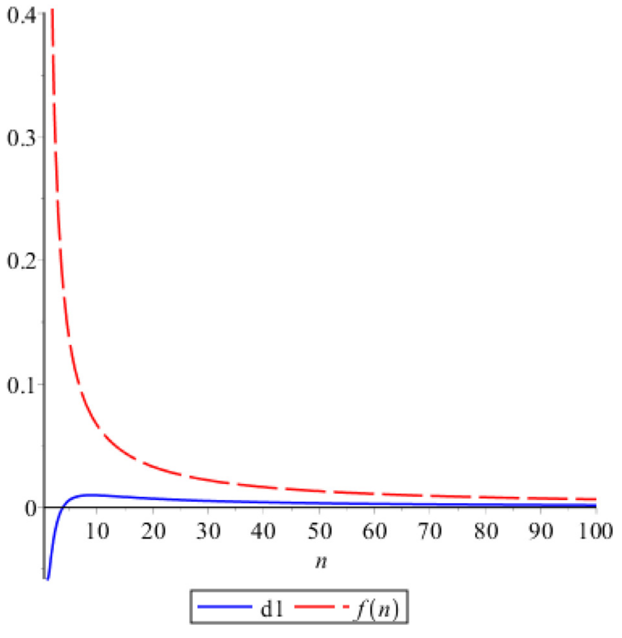

Fig. 2. Comparison between d_1 and $f(n)$.

Case 2: Suppose $\frac{2}{n+1} \le \epsilon < 1$, then $k^0(n,\epsilon) = \hat{k}(n,\epsilon) = \frac{1}{\epsilon}$ and $\pi^0(n,\epsilon) = \hat{\pi}(n,\epsilon) = \epsilon$. Clearly $k_2^*(n,\epsilon) = \frac{1}{\epsilon}$.

Case 3: Suppose $\min(\frac{1}{n+1}, f(n)) < \epsilon < \frac{2}{n+1}$. Consider first the case $\frac{1}{n+1} \le f(n)$. By Lemma 4 $\hat{k}(n,\epsilon) < \frac{1}{\epsilon}$ thus $\hat{\pi}(n,\epsilon) > \epsilon$. Since $k^0(n,\epsilon) = \frac{1}{\epsilon}$ and $\pi^0(n,\epsilon) = \epsilon$, $\hat{\pi}(n,\epsilon) > \pi^0(n,\epsilon)$.

Consider next the case $\frac{1}{n+1} > f(n)$. (i) Suppose $\frac{1}{n+1} \le \epsilon < \frac{2}{n+1}$, then the previous argument applies and $\hat{\pi}(n,\epsilon) > \pi^0(n,\epsilon)$. (ii) Suppose $f(n) < \epsilon < \frac{1}{n+1}$, $k^0(n,\epsilon) = n+1$ and $\hat{k}(n,\epsilon) = \tilde{k}(n,\epsilon)$. We next compare $\pi^0(n,\ \epsilon)$ and $\hat{\pi}(n,\epsilon)$ in this case. First observe that $\tilde{k}(n,\epsilon) < n$ for $f(n) < \epsilon < \frac{1}{n+1}$, thus $\hat{\pi}(n,\epsilon) > n(\pi_1(0,n) - \pi_0(1,n))$. If we can show that

$$n(\pi_1(0,n) - \pi_0(1,n)) > (n+1)\pi_1(n,n+1) \tag{18}$$

then the proof is complete. This is indeed true since (18) holds iff $d_1 \le \epsilon \le d_2$ and we have shown that $d_1 \le f(n)$ and $d_2 \ge \frac{1}{n+1}$. Denote $g(n) = \max(d_1, 0)$, Proposition 3 is complete.

A.4. *Proof of $\pi_1(n-k_1,k) - \pi_0(n-k_1+1,k)$ being increasing in* k_1

Let $m = n - k_1$, we will show that $\pi_1(m,k) - \pi_0(m+1,k)$ is decreasing in m.

$$\pi_1(m,k) - \pi_0(m+1,k) = \frac{(1+(m+1)\epsilon)^2}{(m+k+1)^2} - \frac{(1-k\epsilon)^2}{(m+k+2)^2}$$

The first order condition (using Maple) is

$$\frac{\partial(\pi_1(m,k)-\pi_0(m+1,k))}{\partial m} = G\epsilon^2 + H\epsilon + I$$

where

$$\begin{aligned}
G = &\{2k^5 + 8k^4m + 12k^3m^2 + 8k^2m^3 + 2km^4 + 8k^4 + 30k^3m + 36k^2m^2 \\
&+ 14km^3 + 18k^3 + 54k^2m + 36km^2 + 26k^2 + 40mk + 16k\}/ \\
&\{(m+k+1)^3(m+2+k)^3\} > 0 \\
H = &\{-2k^4 - 8k^3m - 12k^2m^2 - 8km^3 - 2m^4 - 2k^3 - 18k^2m - 30km^2 - 14m^3 \\
&- 36mk - 36m^2 - 12k - 40m - 16\}/ \\
&\{(m+k+1)^3(m+2+k)^3\}
\end{aligned}$$

and

$$I = \frac{-6\,k^2 - 12\,mk - 6\,m^2 - 18\,k - 18\,m - 14}{(m+k+1)^3(m+2+k)^3}$$

Therefore $\frac{\partial(\pi_1(m,k)-\pi_0(m+1,k))}{\partial m}$ is in quadratic in ϵ with $G > 0$.

The equation $\frac{\partial(\pi_1(m,k)-\pi_0(m+1,k))}{\partial m} = 0$ has two solutions in ϵ.

$$\epsilon_1 = -\frac{3\,k^2 + 6\,mk + 3\,m^2 + 9\,k + 9\,m + 7}{k^4 + 4\,k^3m + 6\,k^2m^2 + 4\,km^3 + m^4 + 4\,k^3 + 15\,k^2m + 18\,km^2 + 7\,m^3 + 9\,k^2 + 27\,mk + 18\,m^2 + 13\,k + 20\,m + 8} < 0$$

and

$$\epsilon_2 = \frac{1}{k} > 0$$

Therefore for $0 < \epsilon \le \frac{1}{k}$, $\frac{\partial(\pi_1(m,k)-\pi_0(m+1,k))}{\partial m} < 0$ and $\pi_1(m,k) - \pi_0(m+1,k)$ is decreasing in m. Since $m = n - k_1$, $\pi_1(n-k_1,k) - \pi_0(n-k_1+1,k)$ is increasing in k_1.

A.5. The highest equilirbium payoff of the innovator in UA

The highest equilibrium payoff of the innovator in G_u is

$$\pi_u^*(n,\epsilon) = \begin{cases} (n+1)\pi_1(n,n+1) & \text{if } 0 < \epsilon < g(n) \\ n(\pi_1(0,n) - \pi_0(1,n)) & \text{if } g(n) \le \epsilon \le f(n) \\ \tilde{k}\big(\pi_1(n-\tilde{k},\tilde{k}) - \pi_0(n-\tilde{k}+1,\tilde{k})\big) & \text{if } f(n) < \epsilon < \frac{2}{n+1} \\ \epsilon & \text{if } \frac{2}{n+1} \le \epsilon < 1. \end{cases} \quad (19)$$

where $\tilde{k} = \tilde{k}(n,\epsilon)$.

Proof. Follows immediately from Proposition 3. □

A.6. Equilirbium licensing strategy in NUA

The unique equilibrium licensing strategy of the innovator in G_{nu} is
(i) For $n \geq 3$

$$k_1^{n*}(n,\epsilon) = \begin{cases} n-1 & \text{if } 0 < \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1, \end{cases} \tag{20}$$

$$k_2^{n*}(n,\epsilon) = \begin{cases} \frac{2(n+2\epsilon)}{2n\epsilon+1} - (n-1) & \text{if } 0 < \epsilon \leq \frac{1}{2n-4} \\ 0 & \text{if } \frac{1}{2n-4} \leq \epsilon < 1. \end{cases} \tag{21}$$

(ii) For $n = 2$

$$k_1^{n*}(2,\epsilon) = 1, \tag{22}$$

$$k_2^{n*}(2,\epsilon) = \begin{cases} \frac{3}{4\epsilon+1} & \text{if } 0 < \epsilon \leq \frac{1}{2} \\ \frac{1}{\epsilon} - 1 & \text{if } \frac{1}{2} \leq \epsilon < 1. \end{cases} \tag{23}$$

Proof. By Proposition 1, we focus only on the case where $k_1 + k_2 \leq \epsilon$. The innovator solves

$$\max_{k_1,k_2} \overbrace{k_1(\pi_1(n-k_1,k_1+k_2) - \pi_0(n-k_1,k_1+k_2)) + k_2\pi_1(n-k_1,k_1+k_2)}^{\pi_D}$$

s.t.

$$\begin{aligned} &0 \leq k_1 \leq n-1 \\ &0 \leq k_2 \\ &k_1 + k_2 \leq \frac{1}{\epsilon} \end{aligned} \tag{24}$$

$$\begin{aligned} \pi_D = &-\frac{(k_2\epsilon^2 + 2n\epsilon^2 + 2\epsilon^2)k_1^2}{(n+k_2+1)^2} \\ &-\frac{(k_2^2\epsilon^2 + 2k_2n\epsilon^2 - n^2\epsilon^2 + 2k_2\epsilon^2 - 2n\epsilon^2 - 2n\epsilon - \epsilon^2 - 2\epsilon)k_1}{(n+k_2+1)^2} \\ &+\frac{k_2n^2\epsilon^2 + 2k_2n\epsilon^2 + 2k_2n\epsilon + k_2\epsilon^2 + 2k_2\epsilon + k_2}{(n+k_2+1)^2} \end{aligned}$$

Note first that π_D is continuous on k_1 and k_2. Moreover, for any k_2, π_D is quadratic in k_1. Denote (k_1^*, k_2^*) the optimal choice of the innovator. Given any k_2, let $k_1(k_2)$ be the maximizer of π_D. Then

$$k_1(k_2) = \min\left\{ \underbrace{-\frac{k_2^2\epsilon + (2n\epsilon + 2\epsilon)k_2 - n^2\epsilon - 2n\epsilon - 2n - \epsilon - 2}{2\epsilon(k_2 + 2n + 2)}}_{k_1^s}, n-1, \frac{1}{\epsilon} - k_2 \right\}$$

It can be easily verified that

$$k_1^s < n-1 \quad \text{iff} \quad k_2 > \underbrace{\frac{-2n\epsilon + \sqrt{\epsilon(n^2\epsilon + 2n\epsilon + 2n + 5\epsilon + 2)}}{\epsilon}}_{c_1}$$

and

$$k_1^s < \frac{1}{\epsilon} - k_2 \quad \text{iff} \quad k_2 < \underbrace{\frac{2 - n\epsilon - \epsilon}{\epsilon}}_{c_2}$$

It can also be verified that $c_1 \leq c_2$ iff $\epsilon \leq \frac{1}{2}$. We first analyze the case $0 < \epsilon \leq \frac{1}{2}$. □

Case 1. $0 < \epsilon < \frac{1}{2}$

Subcase 1.1: Suppose $k_2 \leq c_1$, then $n - 1 < k_1^s < \frac{1}{\epsilon} - k_1$ and $k_1(k_2) = n - 1$. Substituting k_1 in π_D with $n - 1$,

$$\pi_D^1 = -\frac{(n-1)\epsilon^2 k_2^2 + (2n^2\epsilon^2 - 4n\epsilon^2 - 2n\epsilon - 2\epsilon^2 - 2\epsilon - 1)k_2 + \epsilon(n-1)(n+1)(n\epsilon - 3\epsilon - 2)}{(n + k_2 + 1)^2}$$

It can be easily verified that $\frac{\partial \pi_D^1}{\partial k_2}$ is decreasing in k_2. Let $\tilde{k}_2$ be the solution of $\frac{\partial \pi_D^1}{\partial k_2} = 0$. Then

$$\tilde{k}_2 = \frac{2n\epsilon + n + 4\epsilon + 1 - 2n^2\epsilon}{2n\epsilon + 1}$$

It can be verified that $\tilde{k}_2 \leq c_1$ iff $\epsilon \leq \frac{1}{2}$ thus π_D is maximized at $k_2 = \tilde{k}_2$ for $k_2 \leq c_1$.

Subcase 1.2: Suppose $c_1 \leq k_2 \leq c_2$, then $k_1^s < n - 1$, $k_1^s < \frac{1}{\epsilon} - k_2$ and $k_1(k_2) = k_1^s$. Substituting k_1 in π_D with k_1^s,

$$\pi_D^2 = \frac{k_2^2\epsilon^2 + (2n\epsilon^2 + 2\epsilon^2)k_2 + n^2\epsilon^2 + 2n\epsilon^2 + 4n\epsilon + \epsilon^2 + 4\epsilon + 4}{4(k_2 + 2n + 2)}$$

It can be verified that π_D^2 is decreasing in k_2 for $0 \leq k_2 \leq c_2$, thus π_D is maximized at $k_2 = c_1$ for $c_1 \leq k_2 \leq c_2$.

Subcase 1.3: Suppose $c_2 \leq k_2$, then $\frac{1}{\epsilon} - k_2 < k_1^s < n - 1$ and $k_1(k_2) = \frac{1}{\epsilon} - k_2$. Since the innovator's payoff is the same for all (k_1, k_2) s.t. $k_1 + k_2 = \frac{1}{\epsilon}$, for any $k_2 \geq c_2$ the innovator obtains the same payoff. By the assumption that the incumbent has the priority in case of a tie, π_D is maximized at $k_2 = c_2$ in the region $k_2 \geq c_2$.

To summarize, for $k_2 \le c_1$, π_D is maximized at $(k_1 = n-1, k_2 = \tilde{k}_2)$; for $k_2 \in [c_1,\ c_2]$, π_D is maximized at $(k_1 = k_1^s, k_2 = c_1)$; for $k_2 \ge c_2$, π_D is maximized at $(k_1 = \frac{1}{\epsilon} - c_2, k_2 = c_2)$. Since π_D is continuous in k_2, π_D is maximized at $k_2 = \max(\tilde{k}_2, 0)$.

For $n \ge 3$, it can be easily verified that $\tilde{k}_2 \ge 0$ iff $\epsilon \le \frac{1}{2n-4}$. Then

$$k_2^*(n,\epsilon) = \begin{cases} \frac{2n\epsilon+n+4\epsilon+1-2n^2\epsilon}{2n\epsilon+1} & \text{if } 0 < \epsilon < \frac{1}{2n-4} \\ 0 & \text{if } \frac{1}{2n-4} \le \epsilon \end{cases}$$

Next we analyze k_1^*. Suppose $0 < \epsilon < \frac{1}{2n-4}$. Since $k_2^* = \tilde{k}_2$ and $\tilde{k}_2 < c_1$ (for $0 \le \epsilon < \frac{1}{2}$), $k_2^* < c_1$. Following the analysis in subcase 1.1, $k_1(k_2^*) = n-1$. Suppose next $\frac{1}{2n-4} \le \epsilon$, $k_1(k_2^*)$ then depends on the relation between 0, c_1 and c_2. (i) If $0 \le c_1$ ($\frac{1}{2n-4} \le \epsilon \le \frac{2}{3n-5}$), following the analysis of subcase 1.1, $k_1(k_2^*) = n-1$. (ii) If $c_1 < 0 \le c_2$ ($\frac{2}{3n-5} \le \epsilon \le \frac{2}{n+1}$), following the analysis of subcase 1.2, $k_1(k_2^*) = k_1^s|_{k_2=0} = \frac{n+1}{4} + \frac{1}{2\epsilon}$. (iii) If $c_2 < 0$ ($\frac{2}{n+1} < \epsilon$), following the analysis of subcase 1.3, $k_1(k_2^*) = \frac{1}{\epsilon}$. Thus for $n \ge 3$

$$k_1^*(n,\epsilon) = \begin{cases} n-1 & \text{if } 0 < \epsilon < \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \le \epsilon \le \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} < \epsilon < \frac{1}{2} \end{cases}$$

$$k_2^*(n,\epsilon) = \begin{cases} \frac{2n\epsilon+n+4\epsilon+1-2n^2\epsilon}{2n\epsilon+1} & \text{if } 0 < \epsilon < \frac{1}{2n-4} \\ 0 & \text{if } \frac{1}{2n-4} \le \epsilon < \frac{1}{2} \end{cases}$$

Consider next $n = 2$. It can be easily verified that $\tilde{k}_2 > 0$. Therefore $k_2^*(2,\epsilon) = \tilde{k}_2$ and $k_2^*(2,\epsilon) < c_1$ (since $\tilde{k}_2 < c_1$ for $0 \le \epsilon < \frac{1}{2}$). Following subcase 1.1 $k_1^*(2,\epsilon) = 1$. The innovator's optimal payoff is obtained for $k_1^* = 1$ and $k_2^* = \tilde{k}_2|_{n=2} = \frac{3}{4\epsilon+1}$.

Case 2. $\frac{1}{2} \le \epsilon < 1$

In this case $c_1 > c_2$, $\tilde{k}_2 > \frac{1}{2}$ and $\tilde{k}_2 > c_1$.

Subcase 2.1: Suppose $k_2 \le c_2$, then $n-1 < k_1^s \le \frac{1}{\epsilon} - k_2$ and $k_1(k_2) = n-1$. Following similar argument as in Subcase 1.1, π_D is maximized at $\min(\tilde{k}_2, c_2)$. Since $\tilde{k}_2 > c_1$ and $c_1 > c_2$, $\tilde{k}_2 > c_2$. Therefore π_D is maximized at $k_1 = n-1$ and $k_2 = c_2$.

Subcase 2.2: Suppose $c_2 \le k_2 \le c_1$, then $k_1^s \ge n-1$, $k_1^s \ge \frac{1}{\epsilon} - k_2$ and $k_1(k_2) = \min(n-1, \frac{1}{\epsilon} - k_2)$. It can be easily verified that $c_2 \le \frac{1}{\epsilon} - n + 1 \le c_1$ for $\epsilon \ge \frac{1}{2}$. (i) Suppose first $c_2 \le k_2 \le \frac{1}{\epsilon} - n + 1$ (or equivalently $n - 1 \le \frac{1}{\epsilon} - k_2$). Then $k_1(k_2) = n-1$. Following similar argument as in Subcase 1.1, π_D is maximized at $\min(\tilde{k}_2, \frac{1}{\epsilon} - n + 1)$. Since $\tilde{k}_2 > c_1 \ge \frac{1}{\epsilon} - n + 1$, π_D is maximized at $(k_1 = n-1, k_2 = \frac{1}{\epsilon} - n + 1)$. (ii) Suppose next $\frac{1}{\epsilon} - n + 1 \le k_2 \le c_1$ (or equivalently $n - 1 \ge \frac{1}{\epsilon} - k_2$) then $k_1(k_2) = \frac{1}{\epsilon} - k_2$. The innovator's payoff is maximized at $k_1 + k_2 = \frac{1}{\epsilon}$.

Subcase 2.3: Suppose $k_2 \ge c_1$, then $\frac{1}{\epsilon} - k_2 < k_1^s < n-1$ and $k_1(k_2) = \frac{1}{\epsilon} - k_2$. The innovator's payoff is maximized again at $k_1 + k_2 = \frac{1}{\epsilon}$.

Consider first $n \geq 3$. Since $\epsilon \geq \frac{1}{2}$, $n-1 \geq \frac{1}{\epsilon}$ holds. Therefore Subcase 2.1 and part (i) of Subcase 2.2 are irrelevant. In this case π_D is maximized at $k_1 + k_2 = \frac{1}{\epsilon}$. By assumption, in this case $k_1^* = \frac{1}{\epsilon}$ and $k_2^* = 0$.

Consider next $n = 2$. Since $\frac{1}{2} \leq \epsilon \leq 1$, $n-1 \leq \frac{1}{\epsilon}$ holds. Therefore Part (ii) of Subcase 2.2 and Subcase 2.3 are irrelevant. Since π_D is continuous on k_2, combining Subcases 2.1 and Part (i) of Subcase 2.2 π_D is maximized at $(k_1 = n-1, k_2 = \frac{1}{\epsilon} - n + 1)$.

A.7. Total number of licensees in NUA

Let $K_{nu}^* = k_1^{n*} + k_2^{n*}$. For $n \geq 3$

$$K_{nu}^*(n,\epsilon) = \begin{cases} \frac{2(n+2\epsilon)}{2n\epsilon+1} & \text{if } 0 < \epsilon \leq \frac{1}{2n-4} \\ n-1 & \text{if } \frac{1}{2n-4} \leq \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1, \end{cases}$$

For $n = 2$

$$K_{nu}^*(n,\epsilon) = \begin{cases} \frac{3}{4\epsilon+1} + 1 & \text{if } 0 < \epsilon \leq \frac{1}{2} \\ \frac{1}{\epsilon} & \text{if } \frac{1}{2} \leq \epsilon < 1 \end{cases}$$

Proof. Follows immediately from A.6. □

A.8. The innovator's equilibrium payoff in NUA

The innovator's equilibrium payoff in G_{nu} is:
For $n \geq 3$

$$\pi_{nu}^*(n,\epsilon) = \begin{cases} \frac{4\epsilon^2+4n\epsilon+1}{4(n+1)} & \text{if } 0 < \epsilon \leq \frac{1}{2n-4} \\ \frac{(n-1)(-(n-3)\epsilon^2+2\epsilon)}{n+1} & \text{if } \frac{1}{2n-4} < \epsilon \leq \frac{2}{3n-5} \\ \frac{(n\epsilon+\epsilon+2)^2}{8(n+1)} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \epsilon & \text{if } \frac{2}{n+1} \leq \epsilon < 1. \end{cases}$$

For $n = 2$

$$\pi_{nu}^*(n,\epsilon) = \begin{cases} \frac{4\epsilon^2+4n\epsilon+1}{4(n+1)} & \text{if } 0 < \epsilon \leq \frac{1}{2} \\ \epsilon & \text{if } \frac{1}{2} \leq \epsilon < 1. \end{cases} \quad (25)$$

Proof. Follows from (5) and A.6. □

A.9. Proof of Proposition 6

If $n = 2$, comparing (19) and (25), it is easy to verify that $\pi_{nu}^*(2,\epsilon) \leq \pi_u^*(2,\epsilon)$ for $0 < \epsilon \leq 1$. We next consider the case $n \geq 3$.

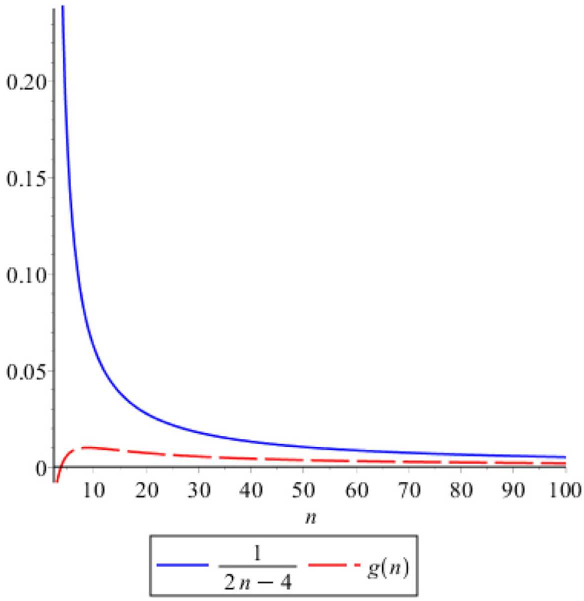

Fig. 3. Comparison between $\frac{1}{2n-4}$ and $g(n)$.

Lemma 5. *Consider the case $n \geq 3$. (i) If $\epsilon \leq g(n)$, $\pi^*_{nu}(n,\epsilon) > \pi^*_u(n,\epsilon)$. (ii) If $\frac{1}{2n-4} \leq \epsilon < \frac{2}{n+1}$, $\pi^*_{nu}(n,\epsilon) < \pi^*_u(n,\epsilon)$. (iii) If $\frac{2}{n+1} \leq \epsilon < 1$, $\pi^*_{nu}(n,\epsilon) = \pi^*_u(n,\epsilon)$.*

Proof. (i) If $\epsilon < g(n)$, in UA the innovator's highest payoff is $\hat{\pi} = (n+1)\pi_1(n, n+1)$ which is obtained when he auctions off $n+1$ licenses and all winners are entrants. In NUA if the innovator chooses $k_1 = 0$ and $k_2 = n+1$ he obtains $\hat{\pi}$. But he can obtain more by choosing other combinations of $(k_1,\ k_2)$. It can be shown that $g(n) < \frac{1}{2n-4}$ for $n \geq 3$ (the analytic proof is difficult, see Fig. 3 for a numerical comparison). Thus by A.6 when $\epsilon < g(n)$, $k^*_1(n,\epsilon) > 0$ and $\pi^*_{nu} > \hat{\pi}$.

(ii) If $\frac{1}{2n-4} \leq \epsilon < \frac{2}{n+1}$ then in NUA $k^*_2(n,\epsilon) = 0$. Since the advantage of NUA on UA lies only on the innovator's ability to charge for a license a higher price to entrants than to incumbent firms, this advantage disappears when $k^*_2(n,\epsilon) = 0$. Moreover, for any $(k_1,\ k_2)$ an incumbent's willingness to pay in NUA is $\pi_1(n-k_1, k_1+k_2) - \pi_0(n-k_1, k_1+k_2)$ while it can be as high as $\pi_1(n-k_1, k_1+k_2) - \pi_0(n-k_1+1, k_1+k_2)$ in UA (incumbent may be willing to pay more to limit entry). Thus $\pi^*_{nu}(n,\epsilon) \leq \pi^*_u(n,\epsilon)$. Since $K^*_{nu} < \frac{1}{\epsilon}$, $\pi^*_{nu}(n,\epsilon) < \pi^*_u(n,\epsilon)$.

(iii) If $\frac{2}{n+1} \leq \epsilon < 1$ the innovator auctions off in total $\frac{1}{\epsilon}$ licenses in both UA and NUA. By Proposition 1, the innovator obtains the same payoff which is the total industry profit ϵ in both auctions. □

Next we focus on the analysis of $g(n) < \epsilon < \frac{1}{2n-4}$. Clearly $\frac{1}{2n-4} < \frac{2}{3n-5} < \frac{2}{n+1}$ for $n \geq 3$. Thus in G_{nu}, $k_1^*(n,\epsilon) = n-1$ and $k_2^*(n,\epsilon) = \frac{2n\epsilon+n+4\epsilon+1-2n^2\epsilon}{2n\epsilon+1}$. In G_u, $\pi_u^*(n,\epsilon)$ depends on whether $\epsilon \leq f(n)$ or $\epsilon > f(n)$.

Case 1: Suppose $f(n) \leq \frac{1}{2n-4}$ (this inequality holds for $n \leq 8$). Then

$$\pi_u^*(n,\epsilon) = \begin{cases} n(\pi_1(0,n) - \pi_0(1,n)) & \text{if } g(n) \leq \epsilon \leq f(n) \\ \tilde{k}\big(\pi_1(n-\tilde{k},\tilde{k}) - \pi_0(n-\tilde{k}+1,\tilde{k})\big) & \text{if } f(n) < \epsilon < \frac{1}{2n-4} \end{cases}$$

We first analyze $g(n) \leq \epsilon \leq f(n)$.

$$\begin{aligned} \pi_{nu}^* - \pi_u^* = {} & \frac{(4n^5+8n^4+4n^3+4n^2+16n+16)\epsilon^2}{4(n+1)^2(n+2)^2} \\ & - \frac{(4n^4+4n^3+8n^2+16n)\epsilon}{4(n+1)^2(n+2)^2} \\ & + \frac{n^3-3n^2-4n+4}{4(n+1)^2(n+2)^2} \end{aligned}$$

It can be easily verified that $\pi_{nu}^* < \pi_u^*$ iff

$$\begin{aligned} & \frac{n^4+n^3+2\,n^2-\sqrt{3\,n^7+14\,n^6+18\,n^5+7\,n^4+24\,n^3+40\,n^2-16}+4\,n}{2(n^5+2\,n^4+n^3+n^2+4\,n+4)} \\ & < \epsilon < \frac{n^4+n^3+2\,n^2+\sqrt{3\,n^7+14\,n^6+18\,n^5+7\,n^4+24\,n^3+40\,n^2-16}+4\,n}{2(n^5+2\,n^4+n^3+n^2+4\,n+4)} \end{aligned}$$

Let $e_1 = \frac{n^4+n^3+2\,n^2-\sqrt{3\,n^7+14\,n^6+18\,n^5+7\,n^4+24\,n^3+40\,n^2-16}+4\,n}{2(n^5+2\,n^4+n^3+n^2+4\,n+4)}$ and

$e_2 = \frac{n^4+n^3+2\,n^2+\sqrt{3\,n^7+14\,n^6+18\,n^5+7\,n^4+24\,n^3+40\,n^2-16}+4\,n}{2(n^5+2\,n^4+n^3+n^2+4\,n+4)}$. Fig. 4 compare the value of $f(n)$, $g(n)$, e_1 and e_2. Note that $n \in [3,\ 8]$ since in this section we deal with $n \geq 3$ and $f(n) \leq \frac{1}{2n-4}$.

It can be easily verified that $g(n)$ and e_1 intersect at $g(n) = e_1 = 0$. Thus $\pi_{nu}^* > \pi_u^*$ for $g(n) \leq \epsilon < e_1$ and $\pi_{nu}^* \leq \pi_u^*$ for $e_1 \leq \epsilon \leq f(n)$.

Next consider the case $f(n) < \epsilon < \frac{1}{2n-4}$. Again, the analytic comparison between π_u^* and π_{nu}^* is difficult and Fig. 5 shows that $\pi_u^*(n,\epsilon) - \pi_{nu}^*(n,\epsilon) \geq 0$ numerically.

To summarize, if $f(n) \leq \frac{1}{2n-4}$, $\pi_{nu}^* > \pi_u^*$ for $g(n) \leq \epsilon < e_1$ and $\pi_{nu}^* \leq \pi_u^*$ for $e_1 \leq \epsilon \leq \frac{1}{2n-4}$.

Case 2: Suppose $f(n) > \frac{1}{2n-4}$ (this inequality holds for $n \geq 9$). Clearly for $g(n) < \epsilon < \frac{1}{2n-4}$, $\pi_u^*(n,\epsilon) = n(\pi_1(0,n) - \pi_0(1,n))$. Again $\pi_{nu}^* < \pi_u^*$ iff $e_1 < \epsilon < e_2$. Fig. 6 shows that $e_2 > \frac{1}{2n-4} > e_1 > g(n)$ numerically. Clearly $\pi_{nu}^* > \pi_u^*$ for $g(n) \leq \epsilon < e_1$ and $\pi_{nu}^* \leq \pi_u^*$ for $e_1 \leq \epsilon < \frac{1}{2n-4}$. Let $h(n) = \max(0, e_1)$, Proposition 6 follows.

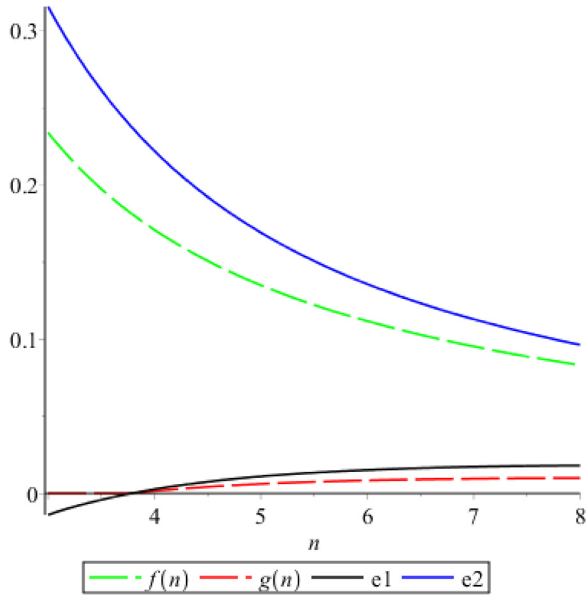

Fig. 4. Comparison between $f(n)$, $g(n)$, e_1 and e_2.

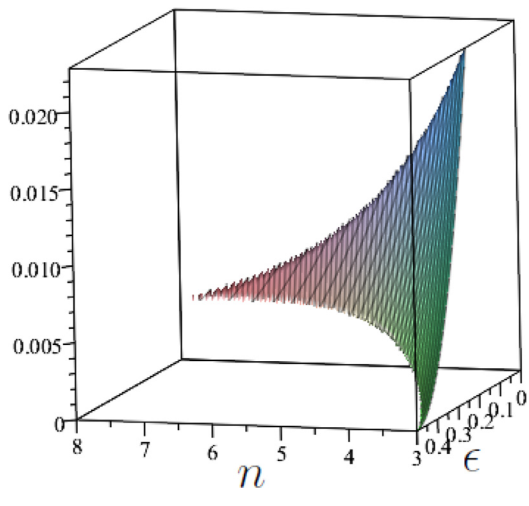

Fig. 5. The value of $\pi_u^*(n,\epsilon) - \pi_{nu}^*(n,\epsilon)$.

A.10. Entry Vs. no entry

Our next goal is to compare our results with the existing literature on optimal licensing where entry is excluded. As shown in previous sections, UA has continuum of equilibrium points and there is no obvious way to predict which equilibrium will emerge. Therefore

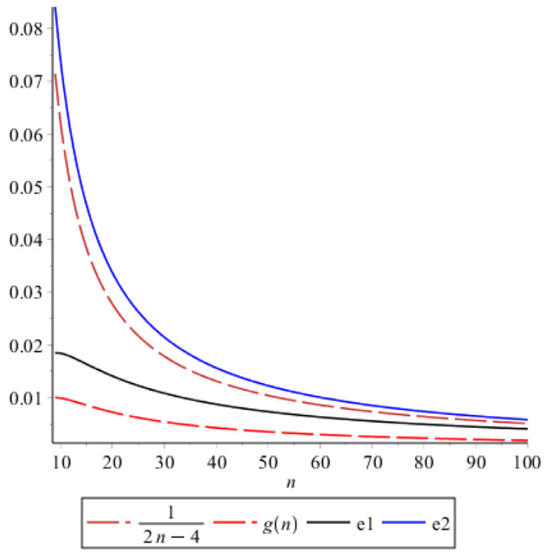

Fig. 6. Comparison between $\frac{1}{2n-4}$, $g(n)$, e_1 and e_2.

we base our study on the comparison between G_0 and G_{nu}, where G_0 is the game defined similarly to G_{nu}, but where entry is excluded.

Suppose bidders do not use dominated strategies. The willingness to pay of each bidder in $G_0(k)$, $k \geq 1$, is uniquely determined and so is the innovator's equilibrium payoff. The next proposition characterizes the innovator's optimal licensing strategy in G_0.

Proposition 7. *The unique equilibrium licensing strategy of the innovator in* G_0 *is:*

(i) For $n \geq 3$

$$k_0^*(n,\epsilon) = \begin{cases} n-1 & \text{if } 0 < \epsilon < \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon < \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1. \end{cases} \tag{26}$$

(ii) For $n \leq 2$

$$k_0^*(n,\epsilon) = n-1.$$

The proof follows from Kamien et al. (1992).

Observe that by Proposition 7, $k_0^*(n,\epsilon) = k_1^{n*}(n,\epsilon)$ where $k_1^{n*}(n,\epsilon)$, the optimal number of incumbent licensees in G_{nu}, is given in A.6. This is not very surprising in light of Corollary 5. By A.6 and Proposition 7 for less significant innovations ($0 < \epsilon \leq \frac{1}{2n-4}$), $k_0^*(n,\epsilon) = k_1^*(n,\epsilon)$ and $k_2^* > 0$. In this case G_{nu} results in a higher diffusion of technology and bigger post-innovation number of firms. The difference in post-innovation number of firms is larger for less significant magnitude of innovation.

The next proposition characterizes the innovator's revenue and the post-innovation market price in G_0.

Proposition 8. *Consider the game G_0. (i) the innovator's equilibrium payoff is*
For $n \geq 3$

$$\pi_0^*(n,\epsilon) = \begin{cases} \frac{(n-1)(-(n-3)\epsilon^2+2\epsilon)}{n+1} & \text{if } 0 < \epsilon \leq \frac{2}{3n-5} \\ \frac{(n\epsilon+\epsilon+2)^2}{8(n+1)} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \epsilon & \text{if } \frac{2}{n+1} \leq \epsilon < 1. \end{cases}$$

For $n \leq 2$

$$\pi_0^*(n,\epsilon) = \frac{(n-1)(-(n-3)\epsilon^2+2\epsilon)}{n+1}.$$

(ii) The post-innovation market price is
For $n \geq 3$

$$p_0^*(n,\epsilon) = \begin{cases} c + \frac{1-(n-1)\epsilon}{n+1} & \text{if } 0 \leq \epsilon \leq \frac{2}{3n-5} \\ c + \frac{2-(n+1)\epsilon}{4(n+1)} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ c & \text{if } \frac{2}{n+1} \leq \epsilon < 1. \end{cases}$$

For $n \leq 2$

$$p_0^*(n,\epsilon) = c + \frac{1-(n-1)\epsilon}{n+1}.$$

Proof. Follows from Proposition 7 □

Corollary 1. *Suppose $n \geq 3$. $\pi_{nu}^*(n,\epsilon) - \pi_0^*(n,\epsilon)$ and $p_0^*(n,\epsilon) - p_{nu}^*(n,\epsilon)$ are both decreasing in n and decreasing in ϵ.*

Proof. See A.11 of the Appendix. □

Corollary 1 asserts that the increment in the innovator's revenue from allowing entry is smaller when there are larger number of incumbent firms or when the magnitude of the innovation is more significant. The same result holds true for the difference in the post-innovation market price. It is shown in Corollary 2 below that these differences vanishes if either n or ϵ is sufficiently large.

Corollary 2. *Allowing entry will not change the innovator's revenue nor the social welfare if either (i) $\epsilon > 0$ and n is sufficiently large, or (ii) $n \geq 3$ and ϵ is sufficiently large.*

Proof. By A.6 and Proposition 7 $k_0^*(n,\epsilon) = k_1^{n*}(n,\epsilon)$ for any n and ϵ, $k_2^{n*}(n,\epsilon) = 0$ for $n \geq 3$ and $\frac{1}{2n-4} \leq \epsilon < 1$. □

Corollary 2 asserts that for any $\epsilon > 0$ there is no difference in price nor in the innovator's payoff between G_{nu} and G_0, for sufficiently large n. This is because the innovator sells licenses to entrants only if he sells licenses to all (but one) incumbent firms (Proposition 5). For a market with large number of incumbents the innovator will not sell licenses to entrants even when entry is allowed. For any $n \geq 3$, the same result holds for sufficiently large ϵ.

Next we characterize the market structure that provides the highest incentive to innovate in G_0.

Proposition 9. *An oligopoly industry with size* $n = \max(3, 2\sqrt{2+\frac{1}{\epsilon}} - 1)$ *maximizes the revenue of the innovator in* G_0.

Proof. See A.12 of the Appendix. □

Proposition 9 asserts that when entry is excluded, the incentive to innovate is maximized when the market is oligopoly and the optimal size is decreasing in the magnitude of the innovation (with at least 3 firms). If however the market is open to entry the incentive to innovate is maximized in a monopoly market (Proposition 4).

A.11. Proof of Corollary 1

(i) For $n \geq 3$

$$\pi^*_{nu}(n,\epsilon) - \pi^*(n,\epsilon) = \begin{cases} \frac{(1-(2n-4)\epsilon)^2}{4(n+1)} & \text{if } 0 \leq \epsilon \leq \frac{1}{2n-4} \\ 0 & \text{if } \frac{1}{2n-4} \leq \epsilon < 1, \end{cases} \tag{27}$$

Let $E = \frac{(1-(2n-4)\epsilon)^2}{4(n+1)}$.

$$\frac{\partial E}{\partial \epsilon} = \frac{((2n-4)\epsilon - 1)(n-2)}{n+1}$$

Observe that (27) is continuous in ϵ and $\frac{\partial E}{\partial \epsilon} \leq 0$ for $0 < \epsilon \leq \frac{1}{2n-4}$. Thus for any $n \geq 3$, $\pi^*_{nu} - \pi^*$ is non-increasing in ϵ for $0 < \epsilon < 1$.

Next observe that for $0 < \epsilon < 1$

$$\pi^*_{nu}(n,\epsilon) - \pi^*(n,\epsilon) = \begin{cases} \frac{(1-(2n-4)\epsilon)^2}{4(n+1)} & \text{if } 3 \leq n \leq \frac{1}{2\epsilon} + 2 \\ 0 & \text{if } \frac{1}{2\epsilon} + 2 \leq n, \end{cases} \tag{28}$$

and (28) is continuous in n. Since

$$\frac{\partial E}{\partial n} = \frac{((2n-4)\epsilon - 1)(2n\epsilon + 8\epsilon + 1)}{4(n+1)^2},$$

$\pi^*_{nu} - \pi^*$ is non-increasing in n for $n \geq 3$.

(ii) For $n \geq 3$

$$p^*(n,\epsilon) - p^*_{nu}(n,\epsilon) = \begin{cases} \frac{1-(2n-4)\epsilon}{2(n+1)} & \text{if } 0 \leq \epsilon \leq \frac{1}{2n-4} \\ 0 & \text{if } \frac{1}{2n-4} \leq \epsilon < 1, \end{cases} \tag{29}$$

Clearly for any $n \geq 3$, $p^*(n,\epsilon) - p^*_{nu}(n,\epsilon)$ is non-increasing in ϵ.

For $0 < \epsilon < 1$,

$$p^*(n,\epsilon) - p^*_{nu}(n,\epsilon) = \begin{cases} \frac{1-(2n-4)\epsilon}{2(n+1)} & \text{if } 3 \le n \le \frac{1}{2\epsilon+2} \\ 0 & \text{if } \frac{1}{2\epsilon} + 2 \le n, \end{cases} \tag{30}$$

Let $F = \frac{1-(2n-4)\epsilon}{2(n+1)}$ It can be easily verified that

$$\frac{\partial F}{\partial n} = -\frac{6\epsilon + 1}{(n+1)^2}$$

Since (30) is continuous in n and $\frac{\partial F}{\partial n} < 0$, (30) is non-increasing in n for $n \ge 3$.

A.12. Proof of Proposition 9

By Proposition 8, for $n \ge 3$

$$\pi_0^*(n,\epsilon) = \begin{cases} \frac{(n-1)(-(n-3)\epsilon^2+2\epsilon)}{n+1} & \text{if } 0 < \epsilon \le \frac{2}{3n-5} \\ \frac{(n\epsilon+\epsilon+2)^2}{8(n+1)} & \text{if } \frac{2}{3n-5} \le \epsilon \le \frac{2}{n+1} \\ \epsilon & \text{if } \frac{2}{n+1} \le \epsilon < 1. \end{cases}$$

For $n \le 2$

$$\pi_0^*(n,\epsilon) = \frac{(n-1)(-(n-3)\epsilon^2 + 2\epsilon)}{n+1}. \tag{31}$$

Consider first $n \ge 3$. First note that for any $\epsilon \ge \frac{1}{2}$, $\pi_0^* = \epsilon$ regardless of the value of n. We next focus on $0 < \epsilon < \frac{1}{2}$.

Subcase 1: Suppose $n < \frac{2}{3\epsilon} + \frac{5}{3}$ (or equivalently $0 < \epsilon < \frac{2}{3n-5}$). Denote

$$\pi_0^{*1} = \frac{\epsilon(n-1)(3\epsilon + 2 - n\epsilon)}{n+1}.$$

It can be easily verified that

$$\frac{\partial \pi_0^{*1}}{\partial n} = \frac{\epsilon(-n^2\epsilon - 2n\epsilon + 7\epsilon + 4)}{(n+1)^2},$$

$$\frac{\partial \pi_0^{*1}}{\partial n} > 0 \quad \text{if} \quad 0 \le n < 2\sqrt{2 + \frac{1}{\epsilon}} - 1$$

and

$$\frac{\partial \pi_D^{*1}}{\partial n} \le 0 \quad \text{if} \quad n \ge 2\sqrt{2 + \frac{1}{\epsilon}} - 1$$

Denote $n^{*1} = 2\sqrt{2 + \frac{1}{\epsilon}} - 1$. It can be easily verified that $3 < n^{*1} < \frac{2}{3\epsilon} + \frac{5}{3}$ for $0 < \epsilon < \frac{1}{2}$. Therefore $3 < n^{*1} < \frac{2}{3\epsilon} + \frac{5}{3}$ for $0 < \epsilon \le \frac{1}{3n-5}$ and n^{*1} is the maximizer of π_0^* for $3 < n < \frac{2}{3\epsilon} + \frac{5}{3}$.

Subcase 2: Suppose $\frac{2}{3\epsilon}+\frac{5}{3}\le n\le\frac{2}{\epsilon}-1$ (or equivalently $\frac{2}{3n-5}\le\epsilon\le\frac{2}{n+1}$). Deonte

$$\pi_0^{*2}=\frac{(n\epsilon+\epsilon+2)^2}{8(n+1)}.$$

It can be easily verified that

$$\frac{\partial\pi_0^{*2}}{\partial n}=\frac{\epsilon^2n^2+2\epsilon^2n+\epsilon^2-4}{8(n+1)^2}$$

and

$$\frac{\partial\pi_0^{*2}}{\partial n}<0\quad\text{for}\quad 0\le n<\frac{2}{\epsilon}-1.$$

Therefore $n^{*2}=\frac{2}{3\epsilon}+\frac{5}{3}$ is the maximizer of π_0^{*2} for $\frac{2}{3\epsilon}+\frac{5}{3}\le n\le\frac{2}{\epsilon}-1$.

Subcase 3: Suppose $\frac{2}{\epsilon}-1\le n$ ($\frac{2}{n+1}\le\epsilon\le\frac{1}{2}$). Then $\pi_0^*(n,\epsilon)=\epsilon$ and the innovator's payoff is the same for any $\frac{2}{\epsilon}-1\le n$.

Combining subcases 1–3, for $n\ge 3$, since π_0^* is continuous in n, $n^*=2\sqrt{2+\frac{1}{\epsilon}}-1$ is the maximizer of π_0^*. Let π_D^* be the innovator's equilibrium payoff when $n=n^*$.

$$\pi_D^*=\begin{cases}\frac{2\epsilon(2\epsilon+1-\sqrt{\epsilon(2\epsilon+1)})(\sqrt{\epsilon(2\epsilon+1)}-\epsilon)}{\sqrt{\epsilon(2\epsilon+1)}} & \text{if } 0<\epsilon<\frac{1}{2}\\ \epsilon & \text{if } \frac{1}{2}\le\epsilon<1\end{cases}$$

Consider next $n=2$. By (31), $\pi_0^*|_{n=2}=\frac{1}{3}\epsilon^2+\frac{2}{3}\epsilon$. Finally consider the case $n=1$. By (31) the innovator obtains 0 since we restrict $k\le n-1$. To provide a more reasonable comparison we assume in this case that the innovator sells the license to the incumbent firm by fixed fee. The innovator's payoff is then $\pi_0^{*'}|_{n=1}=\pi_1(0,1)-\pi_1(1,0)=\frac{1}{4}\epsilon^2+\frac{1}{2}\epsilon$.

Fig. 7 provides the comparison of the innovator's payoff when $n^*=2\sqrt{2+\frac{1}{\epsilon}}-1$, $n=2$ and $n=1$. Clearly in G_0 the innovator obtains the highest payoff in an oligopoly market with $n^*=2\sqrt{2+\frac{1}{\epsilon}}-1$ firms.

A.13. Semi-uniform auction

Finally we introduce and analyze another auction mechanism, a semi-uniform auction (SUA), with a weaker asymmetry requirement than NUA. In this auction the innovator chooses (k_1, k_2), $1\le k_1\le n-1$ and $k_2\ge 0$. The k_1 highest incumbent bidders and the k_2 highest entrant bidders win the auction and all of them pay the same license fee which is the lowest winning bid.[18] Note that the willingness to pay of an incumbent firm is 0 if $k_1=n$. This is the reason we restrict our analysis to $k_1\le n-1$. In SUA, like in NUA, the innovator controls the number of incumbent and the number of entrant licensees, but unlike NUA the innovator charges every licensee the fee.

[18] Note that in SUA the highest losing bid, if submitted by an entrant, may be higher than the willingness to pay of an incumbent winner. To avoid this problem we define the license fee as the lowest winning bid.

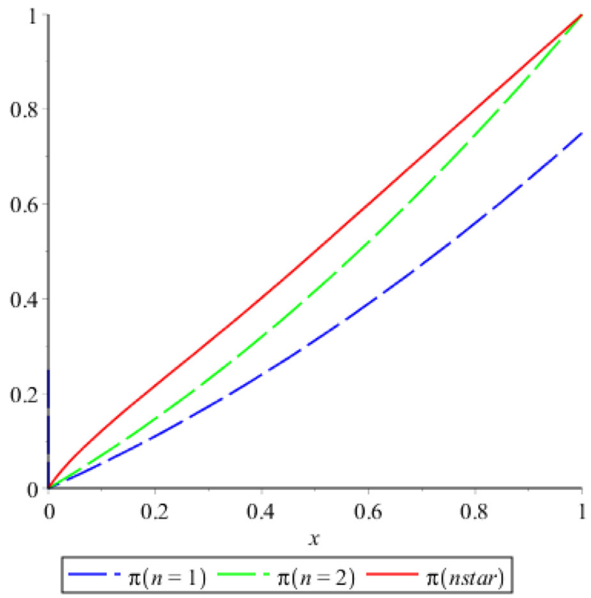

Fig. 7. The innovator's payoff under different n.

Let G_{su} be the game associated with SUA. In the subgame $G_{su}(k_1,\ k_2)$ of G_{su} each incumbent is willing to pay

$$w_l(k_1, k_2) = \pi_1(n - k_1, k_1 + k_2) - \pi_0(n - k_1, k_1 + k_2).$$

Each entrant is willing to pay

$$w_e(k_1, k_2) = \pi_1(n - k_1, k_1 + k_2).$$

Clearly $w_l(k_1, k_2) \leq w_e(k_1, k_2)$. In particular, for $k_1 + k_2 < \frac{1}{\epsilon}$, $\pi_0(n - k_1, k_1 + k_2) > 0$ and $w_l(k_1, k_2) < w_e(k_1, k_2)$. It can be verified that the innovator's equilibrium payoff in $G_{su}(k_1,\ k_2)$ is uniquely determined and it is $(k_1 + k_2)w_l(k_1, k_2)$ for $k_1 > 0$ and $k_2 w_e(0, k_2)$ for $k_1 = 0$.

Remark. Notice that some entrants may bid above the SUA license fee and still do not obtain a license. Yet in equilibrium the innovator has no incentive to increase k_2 since it will increase competition and lower his total revenue.

Let $\pi^*_{su}(n, \epsilon)$ be the innovator's equilibrium payoff in G_{su}.

$$\pi^*_{su}(n, \epsilon) = \max(\pi^0_{su}(n, \epsilon), \hat{\pi}_{su}(n, \epsilon)) \tag{32}$$

where

$$\pi^0_{su}(n, \epsilon) = \max_{k_2 \geq 1} k_2 w_e(0, k_2)$$

and

$$\hat{\pi}_{su}(n,\epsilon) = \max_{\substack{1\le k_1\le n-1\\ 0\le k_2}} (k_1+k_2)w_l(k_1,k_2). \tag{33}$$

When $k_1 = 0$ each entrant licensee pays her entire profit for a license. But when $k_1 > 0$ each entrant licensee pays less, only the willingness to pay of an incumbent licensee.

Proposition 10. *(i) $\pi^*_{su}(n,\epsilon) \le \pi^*_u(n,\epsilon)$ and (ii) $\pi^*_{su}(n,\epsilon) \le \pi^*_{nu}(n,\epsilon)$.*

Proof. (i) In UA the highest equilibrium payoff of the innovator is

$$\pi^*_u(n,\epsilon) = \max(\pi^0_u(n,\epsilon), \hat{\pi}_u(n,\epsilon))$$

where $\pi^0_u(n,\epsilon) = \max_{k\ge 1} kw_e(0,k)$ and $\hat{\pi}_u(n,\epsilon) = \max_{\substack{1\le k_1\le n-1\\ 0\le k_2}}(k_1+k_2)w_h(k_1,k_2)$. Part (i) follows from $w_h(k_1,k_2) \ge w_l(k_1,k_2)$ for any $(k_1,\ k_2)$.

(ii) Follows from the fact that for any $(k_1,\ k_2)$, NUA yields the innovator a higher payoff than SUA. □

The innovator in UA can choose only k while in SUA he can choose in addition the partition of k. In the first glance the innovator should always obtain a higher payoff in SUA than in UA. But this is not necessarily the case. There are cases in which UA yields the innovator a higher payoff than SUA since an incumbent licensee is willing to pay more in UA for further entry prevention. As for the comparison between SUA and NUA, note that in NUA, in addition to choosing the partition $(k_1,\ k_2)$, the innovator can discriminate in price entrants from incumbent licensees. Therefore for any $(k_1,\ k_2)$ and $k_2 > 0$, NUA yields the innovator a higher payoff than SUA.

Next we characterize the market structure that provides the highest incentive to innovate in G_{su}. Like in NUA, in SUA the innovator obtains the highest payoff in a monopoly market.

Proposition 11. *A monopoly industry maximizes the revenue of the innovator if he sells licenses by SUA.*

Proof. The proof is similar to that of Proposition 4, and hence omitted. □

We next analyze for any industry size n the optimal licensing strategy of the innovator in SUA as a function of ϵ. Unlike NUA, the equilibrium licensing strategy is discontinuous for one value of ϵ (the equilibrium revenue of the innovator is however continuous for any ϵ). We will discuss this point after stating the next proposition.

Proposition 12. *Consider the equilibrium of G_{su}. For $n \ge 3$ there exists $r(n)$, $r(n) > 0$, such that (i) if $r(n) < \epsilon \le 1$ then the innovator sells positive number of licenses to entrants only if he sells $n-1$ licenses to all (but 1) incumbent firms. In this region the total number of* *licensees is larger, the less significant is the magnitude of innovation. (ii) At* $\epsilon = r(n)$ *the innovator has two optimal licensing strategies: either selling* $n-1$ *licenses to incumbent firms and some licenses to entrants, or selling* $n+1$ *licenses to only entrants. (iii) if* $0 < \epsilon < r(n)$*, the innovator sells* $n+1$ *licenses to entrants only.*

The proof as well as the exact formula of the equilibrium strategy of the innovator in SUA appears in A.15 of the Appendix.

The reason for selling licenses only to entrant in SUA for less significant innovations is the ability of the innovator to extract the entire industry profit of every entrant licensee. This is in contrast to the case where he sells some licenses also to incumbent firms. In the latter case the license fee an entrant pays is equal to the willingness to pay of an incumbent licensee which decreases to zero as $\epsilon \to 0$. To illustrate this point suppose that $\epsilon = 0$.[19] In this case if the innovator sells some licenses to incumbent firms, every licensee in SUA will pay zero license fee to the innovator. If instead, the innovator sells licenses only to entrants, he obtains the entire industry profit of all new entrant licensees (entrants wouldn't be able to enter the market otherwise). If there are n incumbent firms the linear demand assumption implies that the innovator maximizes his revenue if the number of entrant licensees is $n+1$.

Let us compare the outcome of SUA with the outcome of NUA. First observe that for $\epsilon > r(n)$ in both SUA and NUA the total number of licenses the innovator sells is decreasing in the magnitude of the innovation and the innovator may sell licenses to entrants, only if he also sells licenses to all (but 1) incumbent firms. The main difference between SUA and NUA is when $\epsilon < r(n)$. In this case, unlike NUA, the innovator in SUA sells licenses only to new entrants and not to incumbent firms. This shift in the innovator's optimal strategy generates a discontinuity in the number of licenses at $\epsilon = r(n)$. In contrast, the innovator in NUA can discriminate the entrant licensees and can extract their entire industry profit whether or not he sells licenses to incumbent firms. Therefore in NUA the innovator sells licenses to both new entrants and incumbent firms, even for small ϵ.

Let K^*_{su} and K^*_{nu} be the total number of licenses the innovator sells in SUA and NUA, respectively.

Proposition 13. *Suppose* $n \geq 2$*. There exists* $l(n)$*,* $0 < l(n) < 1$*, such that if* $0 < \epsilon \leq l(n)$*,* $K^*_{nu}(n,\epsilon) > K^*_{su}(n,\epsilon)$*.*

Proof. See A.18 of the Appendix. □

Proposition 13 shows that comparing with SUA, NUA results in higher diffusion of technology for less significant innovations. As shown in Proposition 12 for less significant innovations, the innovator in SUA does not sell licenses to incumbent firms while in

[19] This is the case where the innovator provides no improvement in cost but his technology allows free entry.

NUA he sells licenses to entrants in addition to all (but 1) incumbent firms. Therefore the ability to price discriminate new entrant licensees has positive effect not only on the innovator's revenue but also on social welfare, as compare with SUA.[20]

A.14. The maximizer of $\pi^*_{su}(n,\epsilon)$

Let $(k_1^{s*}(n,\epsilon), k_2^{s*}(n,\epsilon))$ be the maximizer of $\pi^*_{su}(n,\epsilon)$.
For $n \geq 3$

$$k_1^{s*}(n,\epsilon) = \begin{cases} 0 & \text{if } 0 < \epsilon < r(n) \\ n-1 & \text{if } r(n) \leq \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1 \end{cases}$$

and

$$k_2^{s*}(n,\epsilon) = \begin{cases} n+1 & \text{if } 0 < \epsilon < r(n) \\ 2\sqrt{2+\frac{1}{\epsilon}} - (n+1) & \text{if } r(n) \leq \epsilon \leq \frac{4}{n^2+2n-7} \\ 0 & \text{if } \frac{4}{n^2+2n-7} \leq \epsilon < 1. \end{cases}$$

For $n = 2$

$$k_1^{s*}(2,\epsilon) = \begin{cases} 0 & \text{if } 0 < \epsilon < r(2) \\ 1 & \text{if } r(2) \leq \epsilon \leq 1 \end{cases}$$

and

$$k_2^{s*}(2,\epsilon) = \begin{cases} 3 & \text{if } 0 < \epsilon < r(2) \\ 2\sqrt{2+\frac{1}{\epsilon}} - 3 & \text{if } r(2) \leq \epsilon \leq \frac{1}{2} \\ \frac{1}{\epsilon} - 1 & \text{if } \frac{1}{2} \leq \epsilon < 1. \end{cases}$$

For $n = 1$, $k_1^{s*}(1,\epsilon) = 0$ and $k_2^{s*}(1,\epsilon) = 2$.[21]

A.15. Proof of Proposition 12

Let $(0, k_2^0)$ and $(\hat{k}_1, \hat{k}_2)$ be maximizers of $\pi^0_{su}(n,\epsilon)$ and $\hat{\pi}_{su}(n,\epsilon)$, respectively. Clearly either $(0, k_2^0)$ or $(\hat{k}_1, \hat{k}_2)$ is a maximizer of $\pi^*_{su}(n,\epsilon)$.

Lemma 6.

$$k_2^0(n,\epsilon) = \begin{cases} n+1 & \text{if } 0 < \epsilon < \frac{1}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{1}{n+1} \leq \epsilon < 1 \end{cases}$$

Proof. Easy to verify. □

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20 When $\epsilon \leq l(n)$ the market price in SUA is $c + \frac{1-(n+1)\epsilon}{2(n+1)}$ (easy to verify) while the market price in NUA is $c + \frac{1-2\epsilon}{2(n+1)}$ (Proposition A.8).

21 In G_{su} we restrict $k_1 \leq n-1$ therefore $k_1^{s*}(1,\epsilon) = 0$. If, instead, an auction with minimum reservation price is conducted to the monopoly incumbent, there are parameters under which the innovator sells licenses to the incumbent firm in addition to entrants.

Next we focus on the analysis of $\hat{\pi}_{su}(n, \epsilon)$. Note that $n = 1$ does not apply here since $k_1 = 0$ in this case.

Lemma 7. *(i) For* $n \geq 3$

$$\hat{k}_1(n, \epsilon) = \begin{cases} n - 1 & \text{if } 0 < \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1 \end{cases}$$

$$\hat{k}_2(n, \epsilon) = \begin{cases} 2\sqrt{2 + \frac{1}{\epsilon}} - (n+1) & \text{if } 0 < \epsilon \leq \frac{4}{n^2+2n-7} \\ 0 & \text{if } \frac{4}{n^2+2n-7} \leq \epsilon < 1 \end{cases}$$

(ii) For $n = 2$

$$\hat{k}_1(2, \epsilon) = n - 1$$

$$\hat{k}_2(2, \epsilon) = \begin{cases} 2\sqrt{2 + \frac{1}{\epsilon}} - (n+1) & \text{if } 0 < \epsilon \leq \frac{1}{2} \\ \frac{1}{\epsilon} - (n-1) & \text{if } \frac{1}{2} \leq \epsilon < 1 \end{cases}$$

Note that $\frac{4}{n^2+2n-7} \leq \frac{2}{3n-5}$ for $n \geq 3$.

Proof. See A.16 of the Appendix. □

To find the optimal licensing strategy of the innovator, we next compare $\pi^0_{su}(n, \epsilon)$ and $\hat{\pi}_{su}(n, \epsilon)$.

Lemma 8. *For* $n \geq 2$, $\pi^0_{su}(n, \epsilon) > \hat{\pi}_{su}(n, \epsilon)$ *iff* $\epsilon < r(n)$.

The formula of $r(n)$ is quite complicated and it appears in the Appendix A.1.

Proof. See A.17 of the Appendix. □

We are now ready to characterize the optimal licensing strategy of the innovator.

Proposition 14. *For* $n \geq 2$

$$k_1^*(n, \epsilon) = \begin{cases} 0 & \text{if } 0 < \epsilon < r(n) \\ \hat{k}_1(n, \epsilon) & \text{if } r(n) \leq \epsilon < 1 \end{cases}$$

and

$$k_2^*(n, \epsilon) = \begin{cases} n + 1 & \text{if } 0 < \epsilon < r(n) \\ \hat{k}_2(n, \epsilon) & \text{if } r(n) \leq \epsilon < 1. \end{cases}$$

Proof. Follows immediately from Lemmas 6–8. □

A.16. Proof of Lemma 7

We first shows that the innovator in SUA sells licenses to entrants iff he sells licenses to all (but one) incumbent firms.

Lemma 9. *For any $n \geq 2$ and $0 < \epsilon < 1$, $\hat{k}_2(n,\epsilon) > 0$ iff $\hat{k}_1(n,\epsilon) = n-1$.*

Proof. Denote $k = k_1 + k_2$. Suppose first $k = \frac{1}{\epsilon}$. In this case each licensee (entrant or incumbent firm) pays the entire Cournot profit and by the assumption that incumbent firms have priority over entrants in case of a tie, the innovator sells licenses to incumbent firms and only when he exhausts all (but 1) incumbents will he sell licenses to entrants. Suppose next $1 \leq k < \frac{1}{\epsilon}$,

$$\frac{\partial w_l(k_1, k-k_1)}{\partial k_1} = -2\,\frac{\epsilon\,(k\epsilon - 1)}{(n - k1 + k + 1)^2} > 0. \tag{34}$$

For any k, $1 \leq k < \frac{1}{\epsilon}$, the license fee paid by each licensee is increasing in the number of incumbent licensees in k. Therefore the innovator in this case also sells licenses to incumbents first. Lemma 9 follows. □

By Lemma 9, if $k \leq n-1$, $k_1 = k$ and $k_2 = 0$. If, however, $k > n-1$, $k_1 = n-1$ and $k_2 = k - (n-1)$. Therefore

$$\hat{\pi}_{su}(n,\epsilon) = \max\left(\max_{1\leq k\leq n-1} kw_l(k,0), \max_{k_2}((n-1+k_2)w_l(n-1,k_2))\right) \tag{35}$$

Suppose first $n \geq 3$. It can be verified that the maximizer of $\max_{1\leq k\leq n-1} kw_l(k,0)$ is

$$\tilde{k}_1(n,\epsilon) = \begin{cases} n-1 & \text{if } 0 < \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4} + \frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1 \end{cases} \tag{36}$$

and the maximizer of $(n-1+k_2)w_l(n-1,k_2)$ is

$$\bar{k}_2(n,\epsilon) = \begin{cases} 2\sqrt{2+\frac{1}{\epsilon}} - (n+1) & \text{if } 0 < \epsilon \leq \frac{4}{n^2+2n-7} \\ 0 & \text{if } \frac{4}{n^2+2n-7} \leq \epsilon < 1 \end{cases} \tag{37}$$

(37) states that for $\frac{4}{n^2+2n-7} \leq \epsilon < 1$, the innovator is best off selling 0 licenses to entrants even if he sells $n-1$ licenses to incumbent firms. By Lemma 9 $\hat{k}_1(n,\epsilon) = \tilde{k}_1(n,\epsilon)$ and $\hat{k}_2(n,\epsilon) = 0$ in this case. As for $0 < \epsilon \leq \frac{4}{n^2+2n-7}$, the innovator is best off selling positive number of licenses to entrants if he sells $n-1$ licenses to incumbent firms. Since $\frac{4}{n^2+2n-7} \leq \frac{2}{3n-5}$ by (36) the innovator in this case is best off selling $n-1$ licenses to incumbent firms even if $k_2 = 0$. Therefore $\hat{k}_1(n,\epsilon) = n-1$ and $\hat{k}_1(n,\epsilon) = \hat{k}_2(n,\epsilon)$ in this case. Part (i) of Lemma 7 follows.

Suppose next $n = 2$. By Lemma 9, $\hat{k}_1(2,\epsilon) = 1$. It can be easily verified that $\hat{k}_1(2,\epsilon) = \min(2\sqrt{2+\frac{1}{\epsilon}} - 3, \frac{1}{\epsilon} - 1)$.

A.17. Proof of Lemma 8

By Proposition 6 it is easy to verify that

$$\pi^0_{su}(n,\epsilon) \quad = \begin{cases} \frac{(\epsilon(n+1)+1)^2}{4(n+1)} & \text{if } 0 < \epsilon < \frac{1}{n+1} \\ \epsilon & \text{if } \frac{1}{n+1} \le \epsilon < 1 \end{cases} \tag{38}$$

By Proposition 7 it is easy to verify that for $n \ge 3$

$$\hat{\pi}_{su}(n,\epsilon) = \begin{cases} 2\,\epsilon\left(\sqrt{1+2\,\epsilon} - \sqrt{\epsilon}\right)^2 & \text{if } 0 < \epsilon \le \frac{4}{n^2+2n-7} \\ \frac{(n-1)(-(n-3)\epsilon^2+2\epsilon)}{n+1} & \text{if } \frac{4}{n^2+2n-7} < \epsilon \le \frac{2}{3n-5} \\ \frac{(n\epsilon+\epsilon+2)^2}{8(n+1)} & \text{if } \frac{2}{3n-5} \le \epsilon \le \frac{2}{n+1} \\ \epsilon & \text{if } \frac{2}{n+1} \le \epsilon < 1 \end{cases}$$

For $n = 2$

$$\hat{\pi}_{su}(n,\epsilon) = \begin{cases} 2\,\epsilon\left(\sqrt{1+2\,\epsilon} - \sqrt{\epsilon}\right)^2 & \text{if } 0 < \epsilon \le \frac{1}{2} \\ \epsilon & \text{if } \frac{1}{2} \le \epsilon < 1 \end{cases}$$

Suppose $\epsilon \ge \frac{1}{n+1}$, then $\pi^0_{su}(n,\epsilon) = \epsilon$ and $\hat{\pi}_{su}(n,\epsilon) \ge \epsilon$. In this case $\hat{\pi}_{su}(n,\epsilon) \ge \pi^0_{su}(n,\epsilon)$. We next focus on the case $0 < \epsilon \le \frac{1}{n+1}$.

Case 1: Consider first $n \ge 7$. In this case $\frac{1}{n+1} \ge \frac{2}{3n-5}$.

Subcase 1.1: Suppose $\frac{2}{3n-5} \le \epsilon \le \frac{1}{n+1}$.

$$\hat{\pi}_{su}(n,\epsilon) - \pi^0_{su}(n,\epsilon) = -\frac{(n+1)^2\epsilon^2 - 2}{8(n+1)}$$

where $\hat{\pi}_{su}(n,\epsilon) \ge \pi^0_{su}(n,\epsilon)$ iff $-\frac{\sqrt{2}}{n+1} \le \epsilon \le \frac{\sqrt{2}}{n+1}$. Therefore $\hat{\pi}_{su}(n,\epsilon) \ge \pi^0_{su}(n,\epsilon)$ holds for $\frac{2}{3n-5} \le \epsilon \le \frac{1}{n+1}$.

Subcase 1.2: Suppose $\frac{4}{n^2+2n-7} \le \epsilon \le \frac{2}{3n-5}$.

$$\pi^0_{su}(n,\epsilon) - \hat{\pi}_{su}(n,\epsilon) = \frac{\left(5\,n^2 - 14\,n + 13\right)\epsilon^2}{4(n+1)} + \frac{(-6\,n+10)\epsilon}{4(n+1)} + (4\,n+4)^{-1}$$

Note that $\frac{5n^2-14n+13}{4(n+1)} > 0$ for $n \ge 7$. It can be easily verified that $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $\epsilon < \frac{3\,n-5-2\,\sqrt{n^2-4\,n+3}}{5\,n^2-14\,n+13}$ or $\epsilon > \frac{3\,n-5+2\,\sqrt{n^2-4\,n+3}}{5\,n^2-14\,n+13}$.

Denote $e_1 = \frac{3\,n-5-2\,\sqrt{n^2-4\,n+3}}{5\,n^2-14\,n+13}$ and $e_2 = \frac{3\,n-5+2\,\sqrt{n^2-4\,n+3}}{5\,n^2-14\,n+13}$. Fig. 8 compares the value of e_1, e_2, $\frac{4}{n^2+2n-7}$ and $\frac{2}{3n-5}$ numerically. Note that e_1 and $\frac{4}{n^2+2n-7}$ intersects at $n = 16.19$. Thus if $\frac{4}{n^2+2n-7} \le \epsilon \le \frac{2}{3n-5}$, for $7 \le n \le 16.19$, $\hat{\pi}_{su}(n,\epsilon) > \pi^0_{su}(n,\epsilon)$ holds. For $n > 16.19$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $\frac{4}{n^2+2n-7} \le \epsilon < e_1$.

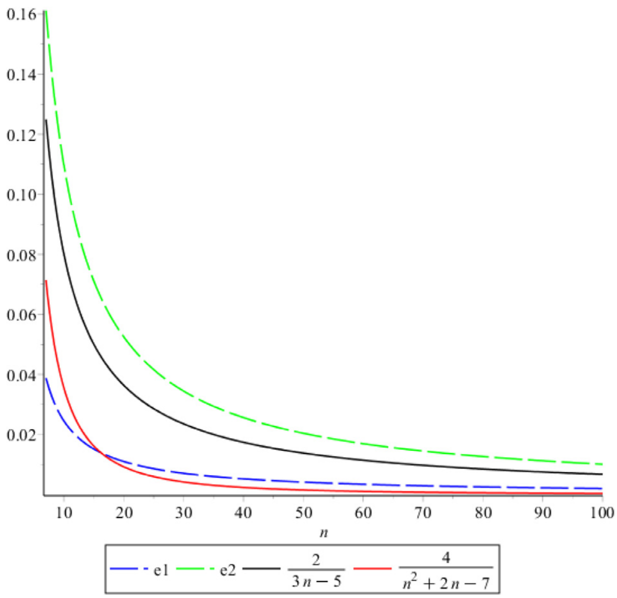

Fig. 8. Comparison between e_1, e_2, $\frac{4}{n^2+2n-7}$ and $\frac{2}{3n-5}$.

Subcase 1.3: Suppose $0 < \epsilon \le \frac{4}{n^2+2n-7}$.

$$\pi^0_{su}(n,\epsilon) - \hat{\pi}_{su}(n,\epsilon) = \frac{n^2\epsilon^2 + \left(16\sqrt{1+2\,\epsilon}\epsilon^{3/2} - 22\,\epsilon^2 - 6\,\epsilon\right)n + 16\sqrt{1+2\,\epsilon}\epsilon^{3/2} - 23\,\epsilon^2 - 6\,\epsilon + 1}{4\,n+4}$$

It can be easily verified that $\pi^0_{su}(n,\epsilon) \le \hat{\pi}_{su}(n,\epsilon)$ iff

$$-\frac{8\sqrt{1+2\,\epsilon}\epsilon^{3/2} - 11\,\epsilon^2 + 2\sqrt{-48\sqrt{1+2\,\epsilon}\epsilon^{7/2} - 12\sqrt{1+2\,\epsilon}\epsilon^{5/2} + 68\,\epsilon^4 + 34\,\epsilon^3 + 2\,\epsilon^2} - 3\,\epsilon}{\epsilon^2} \le n \le \frac{-8\sqrt{1+2\,\epsilon}\epsilon^{3/2} + 11\,\epsilon^2 + 2\sqrt{-48\sqrt{1+2\,\epsilon}\epsilon^{7/2} - 12\sqrt{1+2\,\epsilon}\epsilon^{5/2} + 68\,\epsilon^4 + 34\,\epsilon^3 + 2\,\epsilon^2} + 3\,\epsilon}{\epsilon^2}$$

Denote $f_1 = -\frac{8\sqrt{1+2\,\epsilon}\epsilon^{3/2} - 11\,\epsilon^2 + 2\sqrt{-48\sqrt{1+2\,\epsilon}\epsilon^{7/2} - 12\sqrt{1+2\,\epsilon}\epsilon^{5/2} + 68\,\epsilon^4 + 34\,\epsilon^3 + 2\,\epsilon^2} - 3\,\epsilon}{\epsilon^2}$ and $f_2 = \frac{-8\sqrt{1+2\,\epsilon}\epsilon^{3/2} + 11\,\epsilon^2 + 2\sqrt{-48\sqrt{1+2\,\epsilon}\epsilon^{7/2} - 12\sqrt{1+2\,\epsilon}\epsilon^{5/2} + 68\,\epsilon^4 + 34\,\epsilon^3 + 2\,\epsilon^2} + 3\,\epsilon}{\epsilon^2}$.

Note that for $n \ge 7$, $0 < \epsilon \le \frac{4}{n^2+2n-7}$ iff $n \le 2\sqrt{2+\frac{1}{\epsilon}} - 1$. Fig. 9 shows that $f_2 > 2\sqrt{2+\frac{1}{\epsilon}} - 1$ always holds. Note that ϵ is constraint to $\frac{1}{14}$ since we are dealing in this subcase $\epsilon \le \frac{4}{n^2+2n-7}$ and $n \ge 7$.

Fig. 10 compares the value of f_1 and $2\sqrt{2+\frac{1}{\epsilon}} - 1$. Note that f_1 and $2\sqrt{2+\frac{1}{\epsilon}} - 1$ intersects at $\epsilon = 0.0139$ and $n = 16.19$. By Fig. 10, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff either $\epsilon < 0.0139$ or $\epsilon > 0.0139$ and $n < f_1(\epsilon)$. Or equivalently, when $0 < \epsilon \le \frac{4}{n^2+2n-7}$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff either $n > 16.19$ or $n \le 16.19$ and $\epsilon < f_1^{-1}(n)$ (the existence of $f_1^{-1}(n)$ is shown in Fig. 1).

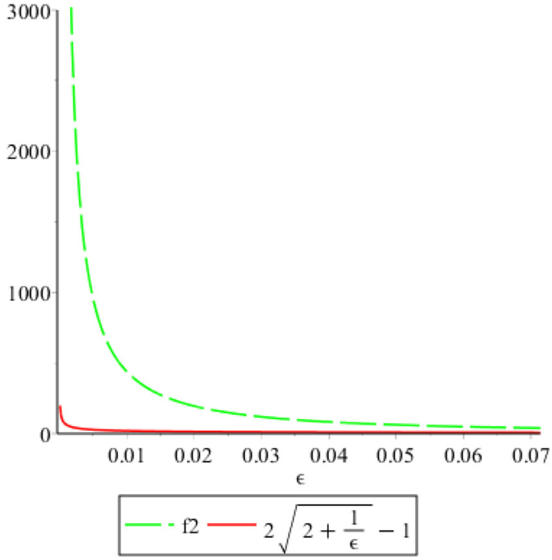

Fig. 9. Comparison between f_2 and $2\sqrt{2+\frac{1}{\epsilon}}-1$.

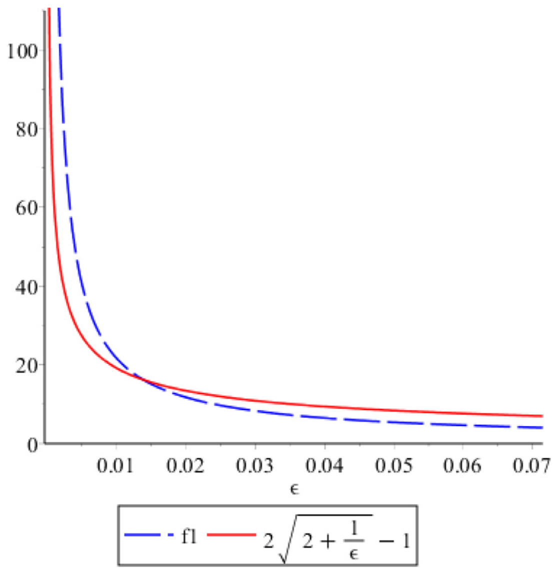

Fig. 10. Comparison between f_1 and $2\sqrt{2+\frac{1}{\epsilon}}-1$.

Combining subcases 1.1–1.3, for $n \geq 17$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $0 < \epsilon < e_1$. For $7 \leq n \leq 16$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $0 < \epsilon < f_1^{-1}(n)$.

Case 2: Consider next $1+2\sqrt{3} \leq n < 7$. In this case $\frac{4}{n^2+2n-7} \leq \frac{1}{n+1} < \frac{2}{3n-5}$.

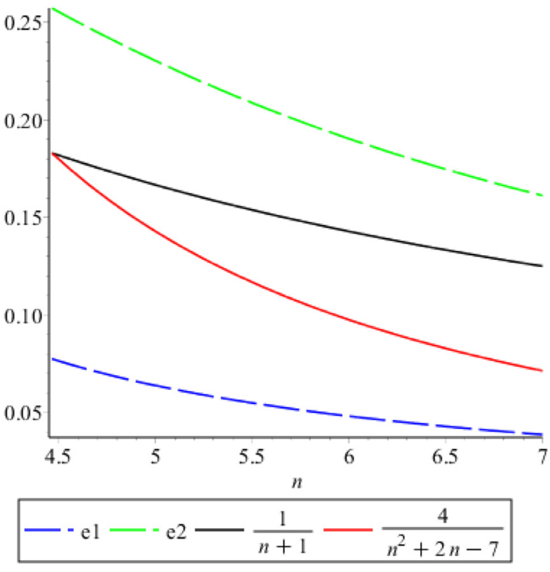

Fig. 11. Comparison between e_1, e_2, $\frac{4}{n^2+2n-7}$ and $\frac{1}{n+1}$.

Subcase 2.1: Suppose $\frac{4}{n^2+2n-7} \le \epsilon \le \frac{1}{n+1}$.

$$\pi^0_{su}(n,\epsilon) - \hat{\pi}_{su}(n,\epsilon) = \frac{(5\,n^2 - 14\,n + 13)\epsilon^2}{4(n+1)} + \frac{(-6\,n + 10)\epsilon}{4(n+1)} + (4\,n + 4)^{-1}$$

where $5\,n^2 - 14\,n + 13 > 0$ for $1 + 2\sqrt{3} \le n < 7$. By the same argument as in Subcase 1.2, $\pi^0_{su}(n,\epsilon) \le \hat{\pi}_{su}(n,\epsilon)$ iff $e_1 \le \epsilon \le e_2$. Fig. 11 shows that $\pi^0_{su}(n,\epsilon) \le \hat{\pi}_{su}(n,\epsilon)$ for $\frac{4}{n^2+2n-7} \le \epsilon \le \frac{1}{n+1}$.

Subcase 2.2: Suppose $0 < \epsilon \le \frac{4}{n^2+2n-7}$, or equivalently $1 + 2\sqrt{3} \le n \le \min(7, \frac{-\epsilon+2\sqrt{2\,\epsilon^2+\epsilon}}{\epsilon})$. Clearly $\epsilon \le \frac{1}{2(1+\sqrt{3})}$. By the same argument as in Subcase 1.3, $\pi^0_{su}(n,\epsilon) \le \hat{\pi}_{su}(n,\epsilon)$ iff $f_1 \le n \le f_2$. It can be easily verified that $f_2 > 7$ for $0 < \epsilon \le \frac{1}{2(1+\sqrt{3})}$. Fig. 12 compares the value of f_1 and $\frac{-\epsilon+2\sqrt{2\,\epsilon^2+\epsilon}}{\epsilon}$. Therefore for $1 + 2\sqrt{3} \le n \le \min(7, \frac{-\epsilon+2\sqrt{2\,\epsilon^2+\epsilon}}{\epsilon})$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff either $0 < \epsilon < f_1^{-1}(7)$ or $f_1^{-1}(7) \le \epsilon$ and $n < f_1$. Or equivalently, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $\epsilon < f_1^{-1}(n)$.

Combining subcases 2.1–2.2, for $1 + 2\sqrt{3} \le n < 7$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $\epsilon < f_1^{-1}(n)$.

Case 3: Suppose $3 \le n \le 1 + 2\sqrt{3}$. In this case $\frac{1}{n+1} \le \frac{4}{n^2+2n-7}$. Consider $0 < \epsilon \le \frac{1}{n+1}$ (or equivalently, $3 \le n \le \min\left(1 + 2\sqrt{3}, \frac{1}{\epsilon} - 1\right)$). Clearly $\epsilon \le \frac{1}{4}$. By the same argument as in Subcase 1.3, $\pi^0_{su}(n,\epsilon) \le \hat{\pi}_{su}(n,\epsilon)$ iff $f_1 \le n \le f_2$. Fig. 13 compares the value of f_1, f_2 and $\frac{1}{\epsilon} - 1$.

Fig. 13 shows that for $3 \le n \le 1 + 2\sqrt{3}$, $\pi^0_{su}(n,\epsilon) > \hat{\pi}_{su}(n,\epsilon)$ iff $\epsilon < f_1^{-1}(n)$.

Finally suppose $n = 2$. Clearly for $\frac{1}{3} \le \epsilon < 1$, $\pi^0_{su}(2,\epsilon) \le \hat{\pi}_{su}(2,\epsilon)$ since $\pi^0_{su}(2,\epsilon) = \epsilon$ and $\hat{\pi}_{su}(2,\epsilon) \ge \epsilon$. For $0 < \epsilon \le \frac{1}{3}$, $\pi^0_{su}(2,\epsilon) > \hat{\pi}_{su}(2,\epsilon)$ iff either $f_1(\epsilon) > 2$ or $f_2(\epsilon) < 2$. It

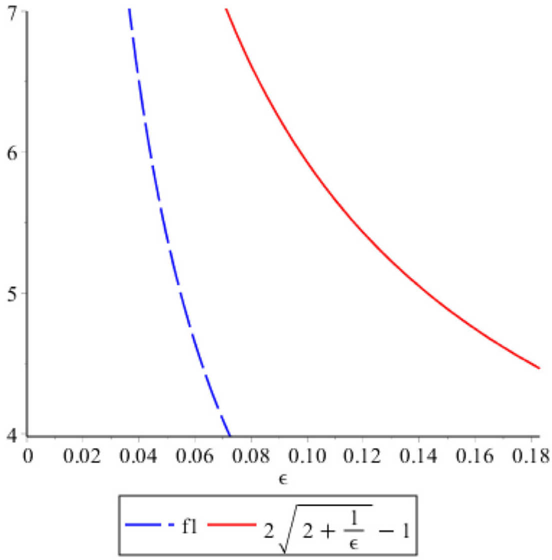

Fig. 12. Comparison between f_1, f_2 and $\frac{-\epsilon+2\sqrt{2\epsilon^2+\epsilon}}{\epsilon}$

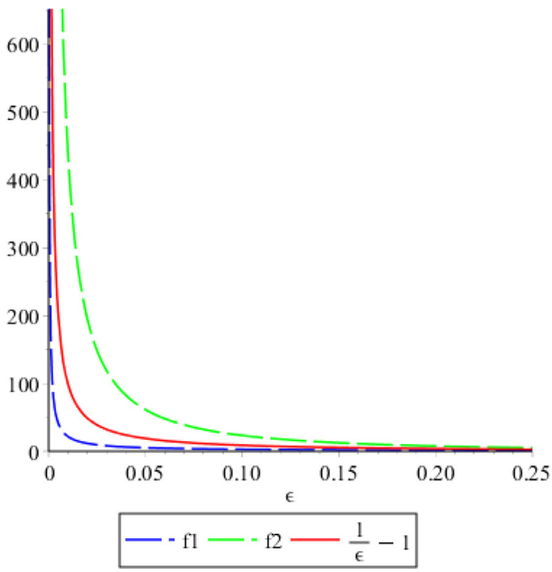

Fig. 13. Comparison between f_1, f_2 and $\frac{1}{\epsilon} - 1$.

can be easily verified that $f_2(\epsilon) > 2$ for any $0 < \epsilon \leq \frac{1}{3}$. Therefore $\pi^0_{su}(2, \epsilon) > \hat{\pi}_{su}(2, \epsilon)$ iff either $f_1(\epsilon) > 2$. Or equivalently, $\pi^0_{su}(2, \epsilon) > \hat{\pi}_{su}(2, \epsilon)$ iff $\epsilon < f_1^{-1}(2)$.

To summarize, for any $n \geq 2$, $\pi^0_{su}(n, \epsilon) \leq \hat{\pi}_{su}(n, \epsilon)$ iff $0 < \epsilon < r(n)$ where $r(n) = e_1$ for $n \geq 16.19$ and $r(n) = f_1^{-1}(n)$ for $1 \leq n < 16.19$.

A.18. Proof of Proposition 13

For $n \geq 3$,

$$K^*_{su}(n,\epsilon) = \begin{cases} n+1 & \text{if } 0 < \epsilon < r(n) \\ 2\sqrt{1+\frac{1}{\epsilon}}-2 & \text{if } r(n) \leq \epsilon \leq \frac{4}{n^2+2n-7} \\ n-1 & \text{if } \frac{4}{n^2+2n-7} \leq \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4}+\frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1 \end{cases}$$

$$K^*_{nu}(n,\epsilon) = \begin{cases} \frac{2(n+2\epsilon)}{2n\epsilon+1} & \text{if } 0 < \epsilon \leq \frac{1}{2n-4} \\ n-1 & \text{if } \frac{1}{2n-4} \leq \epsilon \leq \frac{2}{3n-5} \\ \frac{n+1}{4}+\frac{1}{2\epsilon} & \text{if } \frac{2}{3n-5} \leq \epsilon \leq \frac{2}{n+1} \\ \frac{1}{\epsilon} & \text{if } \frac{2}{n+1} \leq \epsilon < 1 \end{cases}$$

Observe that $K^*_{nu}(n,\epsilon) \xrightarrow{\epsilon \to 0} 2n > n+1$. Since $K^*_{nu}(n,\epsilon)$ is continuous on ϵ and $K^*_{su}(n,\epsilon) = n+1$ for $0 < \epsilon < r(n)$, Proposition 13 follows. The same argument can be applied to case $n = 2$.

References

Arrow, K., 1962. Economic welfare and the allocation of resources for invention. In: The Rate and Direction of Inventive Activity: Economic and Social Factors. Princeton University Press, pp. 609–626.
Chen, Y., Schwartz, M., 2013. Product innovation incentives: monopoly vs. competition. Journal of Economics & Management Strategy 22, 513–528.
Collard-Wexler, A., De Loecker, J., 2014. Reallocation and technology: evidence from the US steel industry. The American Economic Review 105, 131–171.
Gali, E., 2017. Television And The Internet. Mimeo.
Gilbert, R.J., David, M.G.N., 1982. Preemptive patenting and the persistence of monopoly. The American Economic Review 514–526.
Hoppe, H.C., Jehiel, P., Moldovanu, B., 2006. License auctions and market structure. Journal of Economics & Management Strategy 15.2 371–396.
Kamien, M.I., 1992. Patent licensing 1, 331–354.
Kamien, M.I., Oren, S.S., Tauman, Y., 1992. Optimal licensing of costreducing innovation. Journal of Mathematical Economics 21 (5), 483–508.
Kamien, M.I., Tauman, Y., 1986. Fees versus royalties and the private value of a patent. The Quarterly Journal of Economics 471–491.
Katz, M.L., Shapiro, C., 1985. On the licensing of innovations. The RAND Journal of Economics 504–520.
Sen, S., Tauman, T., 2007. General licensing schemes for a cost-reducing innovation. Games and Economic Behavior 59, 163–186.
Schumpeter, J.A., 1942. Socialism, Capitalism and Democracy. Harper and Brothers.
Tauman, Y., Weiss, Y., Zhao, C., 2017. Bargaining in Patent Licensing with Inefficient Outcomes. Mimeo.